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BACK-SCATTERING OF ELECTROMAGNETIC WAVES FROM SPHERES AND SPHERICAL SHELLS

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Geophysics Research Division
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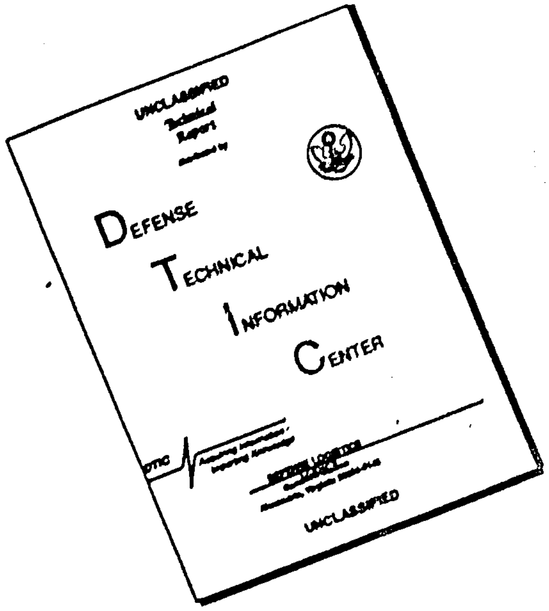

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ABSTRACT

The classical theory of the scattering of a plane electromagnetic wave by a sphere is reviewed, and the difficulties in getting numerical answers from this formal solution are discussed. It is shown how the computations may be simplified by using suitably defined logarithmic derivative functions and the recurrence formulas due to Infeld. By this method, numerical answers for the scattering amplitude coefficients of any order can be computed exactly, even if the index of refraction is complex.

The technique mentioned above has been used to determine the scattering amplitude coefficients and the back-scattering cross section for one special case involving water spheres with sizes comparable to the wavelength. The theoretical results are compared with those obtained experimentally.

A rigorous solution is also given for the scattering of a plane electromagnetic wave from two concentric spheres of different dielectric constant. This problem is formulated in a manner similar to that for a single sphere, and the scattering amplitude coefficients are expressed in terms of spherical Bessel functions and the logarithmic derivative functions. The application to a particular physical problem is indicated.

ACKNOWLEDGMENTS

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CONTENTS

ILLUSTRATIONS

GLOSSARY

Symbol	*Name*	*Units*
a	Radius of the sphere	meter
a_n', b_n'	Scattering amplitude coefficients	dimensionless
b	Radius of the spherical shell	meter
$\mathbf{B}$	Fundamental magnetic vector	volt-second/square meter
$D_n(x)$	General Bessel density function	
$\mathbf{E}$	Fundamental electric vector	volt/meter
$H_n^{(2)}(x)$	Hankel function of the second kind	
$\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$	Roman letters indicate unit vectors along the x, y and z axes	
j	Square root of minus one	
$J_n(x)$	Bessel function of the first kind	
k	Wave number of space	radian/meter
k_1	Complex wave number of the sphere	radian/meter
k_2	Complex wave number of the spherical shell	radian/meter
$\mathbf{l}, \mathbf{m}, \mathbf{n}$	Orthogonal spherical vector wave functions	
m, n	Integer indices	
N	Complex index of refraction of the sphere	dimensionless
N_1	Complex index of refraction of the sphere (Section 5)	dimensionless
N_2	Complex index of refraction of the spherical shell	dimensionless
$P_n^m(\cos\theta)$	Associated Legendre polynomial	
P_a	Total absorbed power	watt
P_s	Total scattered power	watt
P_t	Total power removed from the incident wave	watt
Q_a	Total absorption cross section	square meter
Q_s	Total scattering cross section	square meter
Q_t	Total attenuation cross section	square meter
r	Spherical coordinate	meter
$R_n(x)$, $S_n(x)$	Bessel density functions	
$\mathbf{S}$	Complex Poynting vector	watt/square meter
v_c	Characteristic velocity of space	meter/second
v_1	Complex characteristic velocity of the sphere	meter/second
v_2	Complex characteristic velocity of the spherical shell	meter/second
W_i	Power density incident upon the scatterer	watt/square meter
W_s	Power density of the far zone scattered field in direction of source	
x, y, z	Cartesian coordinates	meter
$z_n^{(p)}(x)$	General spherical Bessel function	
$Z_n(x)$	General cylindrical Bessel function	
α	Electrical size of the sphere	radian
ν	Electrical size of the spherical shell	radian
$\delta_n(x)$	General logarithmic derivative function	
ϵ_0	Dielectric constant of space	farad/meter

GLOSSARY (*Continued*)

Symbol	*Name*	*Units*
ϵ_e	Real effective absolute dielectric constant	farad/meter
ζ_0	Characteristic impedance of space	ohm
θ, ϕ	Spherical coordinates	radian
λ	Wavelength	meter
μ_0	Permeability of space	henry/meter
ν_0	Magnetic constant of space	meter/henry
ξ	Complex dielectric factor	farad/meter
ξ_1	Complex dielectric factor of the sphere (Section 5)	farad/meter
ξ_2	Complex dielectric factor of the spherical shell	farad/meter
ξ_r	Relative complex dielectric factor	dimensionless
$\rho_n(x), \sigma_n(x)$	Logarithmic derivative functions	
σ_e	Real effective conductivity	mho
σ	Back-scattering cross section	square meter
Ω	Solid angle	steradian
ω	Angular frequency	radian/second
Subscript i	Refers to the incident field	
Subscript s	Refers to the secondary field outside the scatterer	
Subscript t	Refers to the secondary field inside the scatterer	
$e^{j\omega t}$	Assumed time dependence	

BACK-SCATTERING OF ELECTROMAGNETIC WAVES FROM SPHERES AND SPHERICAL SHELLS

1. INTRODUCTION

When any object is placed in the field of an electromagnetic wave it removes energy from the field. In general, some of the energy is dissipated internally as heat, and some of the energy is reradiated to produce a secondary or scattered field. One of the important problems of electrodynamics is to determine the amount of energy absorbed and the amount and distribution of the scattered energy for a particular scattering object illuminated under specified conditions. One of the simplest problems of this type to visualize is that of a plane wave incident upon a spherical particle. Although the geometry of this problem is simple, the problem itself is very important and has received considerable attention from investigators throughout the years.

Pioneering in this field, Tyndall (1869a) showed experimentally the relation of the size of suspended spherical particles to the intensity and polarization of scattered light. Although he was mainly interested in studying the chemical action of light, he discussed the problem of the color and polarization of scattered light in several additional papers (1869b, 1870). Shortly after Tyndall's work, Lord Rayleigh (1871, 1881, 1897, 1899) published a series of papers which became the basis for the classical theory of scattering. He handled the case of particles very much smaller than the wavelength of light and established his famous result that under these conditions the scattering is inversely proportional to the fourth power of the wavelength.

The first thorough treatment for a spherical particle of any size and any electrical properties was given by Mie (1908). His solution is an infinite series of spherical-mode functions with amplitude coefficients that are determined from the boundary conditions at the surface of the sphere. Each of these spherical-mode functions itself involves infinite series, so that in the general case of a large sphere with a complex index of refraction the problem of getting numerical answers is quite involved. However, the Mie paper still remains the fundamental contribution to the subject.

Since publication of the paper by Mie, additional investigations have been made by Debye (1909), Gans and Happel (1909), Bromwich (1920), Ray (1921), Jobst (1925), Blumer (1925, 1926), Stratton et al. (1930, 1931, 1941), Trinks (1935), Engelhard and Friess (1937), Ryde et al. (1941, 1944, 1945, 1946), Ruedy (1941, 1943, 1944), LaMer et al. (1943, 1946, 1948), Brillouin (1943, 1944, 1949), L. Goldstein (1945), Sinclair (1947), Houghton et al. (1949), Klotzbaugh and Duckett (1949) and others. The complete theory is given concisely by Stratton (1941) and L. Goldstein (1945). This will be reviewed in Section 3.

It is not the present purpose to review all the papers dealing with the scattering problem. Instead, the discussion will be restricted to one particular aspect of the problem.

With the development of microwave radar during World War II, it was found that, for sufficiently small wavelengths, rain could produce an appreciable echo and substantial attenuation. The echo phenomenon, was important for several reasons. For the usual tactical use of the equipment, it was important to distinguish between atmospheric reflections and operational targets. Moreover, as a meteorological tool, the observance of echoes was very useful in mapping the rain areas and in showing their movements. The problem of attenuation was important to both radar and communication. It is not surprising, then, that these problems received considerable attention.

To investigate theoretically the microwave reflection from rain, it is necessary to assume that the drops are randomly distributed in space and that the mutual interaction between drops is negligible. The first assumption appears to be valid without further comment. The second assumption is based on Trink's (1935) analysis that for Rayleigh scattering the mutual interaction is negligible for sphere spacings greater than two or three sphere diameters, plus the fact that in actual rain the average spacing between drops is many times this value (Palmer (1949)). Under these conditions the problem can be divided into two parts: (1) finding the drop-size distribution and (2) finding the reflection from a single drop. The drop-size distribution is strictly a meteorological problem; the reflection from a single drop is a problem in electrodynamics.

The first thorough investigation of the reflection and attenuation effects from rain was made by Ryde et al. (1941, 1944, 1945, 1946). His method was to compute the numerical results for a single drop using the Mie theory and to use the assumptions listed above to make application to the case of many drops. The main difficulty lay in getting the numerical results from the Mie theory. Despite the tremendous labor involved, however, he computed numerical results for a variety of cases. For radars of longer wavelength, it was found that the Rayleigh theory was adequate, but for radars of shorter wavelength, a more exact theory was needed. The work of Ryde is discussed briefly by L. Goldstein (1945), who has extended it somewhat, using more detailed meteorological data. Recently a table of scattering functions has been prepared by Lowan (1949) and these results have been used by Haddock (1947) in a further contribution to this problem.

In all cases, a major difficulty has been the evaluation of the reflection from a single sphere. It was an interest in this problem that prompted part of the present investigation.

In Section 2, the common absorption and scattering parameters are defined. Particular attention is given to the back-scattering cross section, which is the main subject of this report.

The classical theory of scattering from a sphere is reviewed in Section 3. The difficulties in getting numerical answers are discussed, and it is shown how the computations may be simplified by using logarithmic derivative functions and the recurrence formulas due to Infeld (1947). This method is used to evaluate the back-scattering cross sections for water spheres at $\lambda = 16.230$ cm in the size range from $0.6 < 2\pi a/\lambda < 6$.

In Section 4 a method for measuring the back-scattering from water spheres is described briefly. Comparison is made of the experimental and the theoretical results.

Another physical problem of interest is the reflection from a melting snowflake or ice particle. As a first approximation, one may consider, as the mathematical model corresponding to this phenomenon, the reflection from two concentric spheres with different complex dielectric factors. Using the same method as for a single sphere, it is possible to determine a rigorous formal solution for this scattering. This is done in Section 5.

2. CROSS SECTIONS FOR SCATTERING AND ABSORPTION

2.1. TOTAL CROSS SECTIONS

The concept of scattering and absorption parameters is found in many branches of physics. The most commonly used parameters are those related to the total power scattered and the total power absorbed by

an object in the field of an electromagnetic wave. They may be defined as follows: Let

W_i = the power density incident upon the object,
P_s = the total scattered power,
P_a = the total absorbed power,
P_t = the total power removed from the incident wave
= $(P_s + P_a)$.

Then the total scattering cross section is

$$Q_s = \frac{P_s}{W_i}, \tag{2.1}$$

the total absorption cross section is

$$Q_a = \frac{P_a}{W_i} \tag{2.2}$$

and the total attenuation cross section is

$$Q_t = \frac{P_t}{W_i} = Q_s + Q_a. \tag{2.3}$$

2.2. THE BACK-SCATTERING CROSS SECTION

The back-scattering cross section σ is a lumped measure of the ability of a scattering obstacle to reradiate energy in the direction of the source. Under the assumption of a plane electromagnetic wave incident upon the scatterer, this parameter can be determined from a knowledge of the far-zone scattered field. It is commonly defined as follows:

$$\sigma = 4\pi r^2 \frac{W_s}{W_i} \tag{2.4}$$

where

W_i = the power density in the plane wave incident upon the scatterer,
W_s = the power density of the far zone scattered field in the direction of the source and
r = the distance from the scatterer at which the evaluation of W_s is made.

It should be noted that for an isotropic scatterer, the back-scattering cross section σ is equal to the total scattering cross section Q_s as defined by Eq. (2.1). In fact, σ may be defined in general as the total scattering cross section of a fictitious isotropic scatterer which scatters energy in all directions with intensity equal to that scattered directly back toward the source by the actual scattering object.

Equation (2.4) may be written in alternative equivalent forms by writing W_s and W_i more explicitly. Thus W_s is given formally by

$$W_s = \frac{dP_s}{dA} \tag{2.5}$$

where dA is an element of area located a distance r from the scatterer in the direction of the source. In addition, $dA = r^2 d\Omega$, where $d\Omega$ is an element of solid angle measured from the scatterer. Using this,

$$W_s = \frac{1}{r^2}\frac{dP_s}{d\Omega} \tag{2.6}$$

and

$$\sigma = \frac{4\pi}{W_i}\frac{dP_s}{d\Omega}, \tag{2.7}$$

where $dP_s/d\Omega$ is evaluated in the direction of the source.

Since the radiation incident upon the scattering object is assumed to be a plane wave,

$$W_i = \frac{E_0^2}{2\zeta_0}, \tag{2.8}$$

where E_0 is the magnitude of the electric field in the plane wave and ζ_0 is the characteristic impedance of space. Using Eq. (2.8), Eq. (2.7) becomes

$$\sigma = \frac{8\pi\zeta_0}{E_0^2}\frac{dP_s}{d\Omega}. \tag{2.9}$$

At any point in the far zone from the scatterer, the scattered field is also essentially a plane wave. Therefore, in terms of the far-zone scattered field E_s,

$$W_s = \frac{|E_s|^2}{2\zeta_0}. \tag{2.10}$$

Substituting Eqs. (2.8) and (2.10) into Eq. (2.4) yields

$$\sigma = 4\pi r^2\left|\frac{E_s}{E_0}\right|^2. \tag{2.11}$$

Finally, if the far-zone scattered field is referred back to unit range — i.e., let $E_{s_1} = rE_s$ — then Eq. (2.11) becomes

$$\sigma = 4\pi\left|\frac{E_{s_1}}{E_0}\right|^2. \tag{2.12}$$

All of these equivalent forms are found in the literature.

2.3. RESTRICTIONS ON THE CROSS SECTIONS

In the preceding sections, several cross sections have been introduced as convenient parameters for indicating the scattering and absorbing properties of material objects. So long as this is all that is implied, no difficulties arise. However, they are often associated with the arbitrary picture that the Poynting vector is a unique measure of the power flow through space. The pitfalls of this incorrect picture are pointed out by King (1945). It should be noted that although all the cross sections defined have the dimensions of an area, none of these "areas" is in general equal to the physical area of the object. The value of any one of the cross sections depends in general on the wavelength, polarization and angle of incidence of the impinging radiation, as well as on the dimensions, shape and material of the object.

3. THE THEORY OF THE SCATTERING OF A PLANE WAVE BY A SPHERE

3.1. INTRODUCTION

The general problem of a plane wave scattered by a spherical particle was first studied in detail by Mie (1908). He treated the problem rigorously according to classical electrodynamics. Briefly, his method is as follows: The Maxwell field equations are written in spherical coordinates. These equations are simplified by resolving the electromagnetic field into transverse electric (TE) and transverse magnetic (TM) waves, and particular elementary spherical-wave solutions of TE and TM type are found. The complete

solution for the scattered field is then written formally as the sum of all the elementary solutions with unknown amplitude coefficients. The incident plane wave is also expressed in terms of a linear combination of the elementary spherical-wave solutions. Finally, the initially undetermined coefficients are evaluated by applying the boundary conditions at the surface of the sphere. This completes the formal solution.

The complete solution to this problem is given concisely by Stratton (1941). Following Hansen (1935), he develops a set of orthogonal spherical vector wave functions, each of which is a solution to the vector wave equation. The incident plane wave is expanded in terms of these functions, and the scattered field is expressed formally as a similar expansion with unknown amplitude coefficients. As in the Mie solution, the unknown coefficients are determined by applying the boundary conditions at the surface of the sphere. The method of Stratton is also given by L. Goldstein (1945), who shows the relation between the coefficients of Stratton and those of Mie.

In the following sections the method of Stratton is reviewed following essentially the notation of Goldstein. The difficulties of obtaining numerical answers for the case of a sphere with a complex dielectric constant are discussed, and it is shown how logarithmic derivative functions and the recurrence formulas due to Infeld (1947) may be used to simplify the computations. Finally, this method is used to obtain quantitative results for one special case involving water spheres.

3.2. FORMULATION OF THE PROBLEM

Consider a sphere of radius a, complex dielectric factor ξ and permeability μ_0 which is isolated in free space.* The interior of the sphere is called region 1; the surrounding space is called region 2. The center of the sphere is chosen as the origin of a rectangular coordinate system. A plane electromagnetic wave, with the electric field parallel to the x axis, is propagated along the z axis and strikes the sphere. Periodic time dependence of the form $e^{j\omega t}$ is assumed † but in general will not be written explicitly. MKS units are used throughout. Quantities with bold letters represent space vectors; accented quantities in roman letters — e.g., $\hat{\mathrm{i}}, \hat{\mathrm{j}}, \hat{\mathrm{k}}, \hat{\mathrm{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ — represent unit vectors.

The incident plane wave may be expressed as

$$\mathbf{E}_i = \hat{\mathrm{i}}E_x = \hat{\mathrm{i}}E_0 e^{-jkz} \tag{3.1}$$

$$\mathbf{B}_i = \hat{\mathrm{j}}B_y = \hat{\mathrm{j}}\frac{E_0}{v_0}e^{-jkz} \tag{3.2}$$

where

$$k = \omega(\mu_0\epsilon_0)^{\frac{1}{2}} = \frac{2\pi}{\lambda} \tag{3.3}$$

is the wave number of space and

$$v_0 = (\mu_0\epsilon_0)^{-\frac{1}{2}} = \frac{\omega}{k} \tag{3.4}$$

is the characteristic velocity of space.

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* This is a special case of the more general problem treated by Mie and Stratton where the surrounding medium may also have complex electrical properties. However, the method of solution is identical.

† Stratton assumes $e^{-i\omega t}$ time dependence. Therefore, in comparing equations written here with those of Stratton replace $+j$ by $-i$.

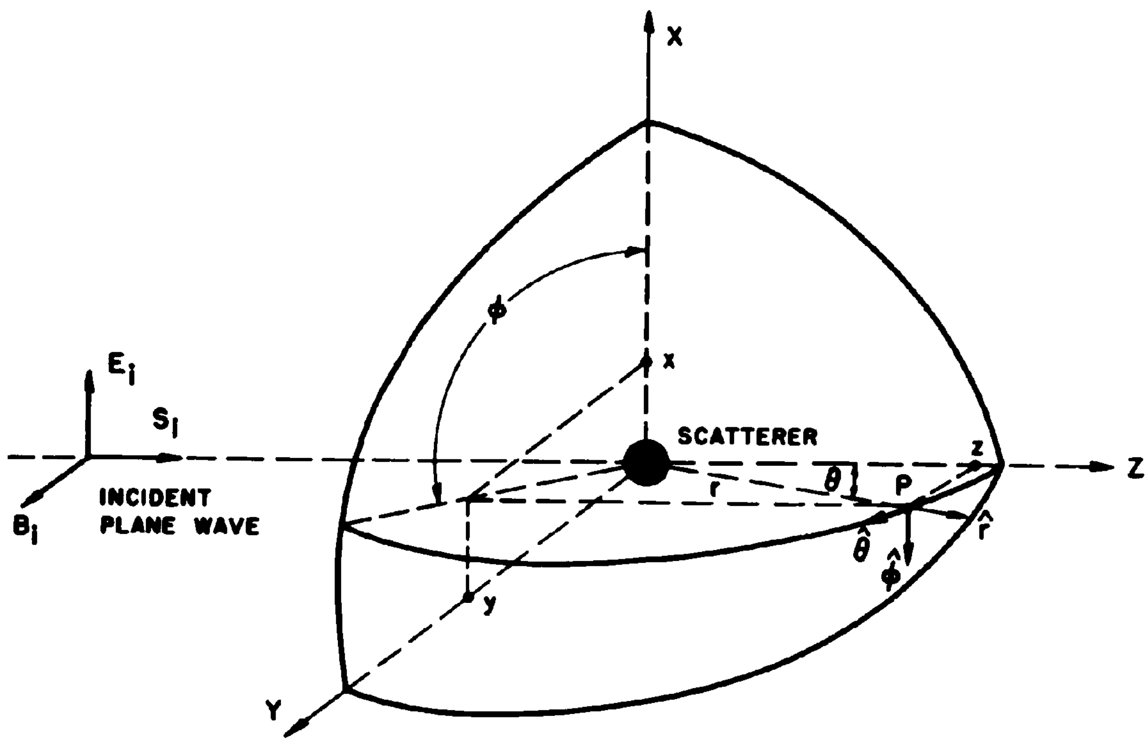

FIG. 1. Spherical coordinate system.

The orthogonal spherical vector wave functions are obtained as follows: The scalar wave equation in spherical coordinates is solved by separation of variables to yield

$$f^{\beta}_{\substack{e \\ o}mn} = \frac{\cos}{\sin} m\phi \quad P_n^m(\cos\theta)\, z_n^{(\beta)}(kr). \tag{3.5}$$

Here, $P_n^m(\cos\theta)$ is the associated Legendre polynomial of the first kind, nth degree and mth order; $z_n^{(\beta)}(kr)$ is the spherical Bessel function defined by

$$z_n^{(1)}(kr) = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(kr) \tag{3.6}$$

$$z_n^{(2)}(kr) = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(kr). \tag{3.7}$$

$J_{n+\frac{1}{2}}(kr)$ is the Bessel function of the first kind and half-integer order; $H_{n+\frac{1}{2}}^{(2)}(kr)$ is the Hankel function of the second kind and half-integer order. The subscripts e and o refer to the even and odd ϕ dependence. The spherical vector wave functions are defined as

$$\mathbf{l}^{\beta}_{\substack{e \\ o}mn} = \nabla f^{\beta}_{\substack{e \\ o}mn} \tag{3.8}$$

$$\mathbf{m}^{\beta}_{\substack{e \\ o}mn} = \nabla \times \left(\mathbf{r} f^{\beta}_{\substack{e \\ o}mn}\right) \tag{3.9}$$

$$\mathbf{n}^{\rho}_{\substack{e\\o}mn} = \frac{1}{k}\nabla \times \mathbf{m}^{\rho}_{\substack{e\\o}mn}. \tag{3.10}$$

Carrying out the indicated operations yields

$$\begin{aligned} \mathbf{l}^{\rho}_{\substack{e\\o}mn} = {} & \frac{\partial}{\partial r} z_n^{(\rho)}(kr) \quad P_n^m(\cos\theta) \frac{\cos}{\sin} m\phi\, \hat{\mathbf{r}} \\ & + \frac{1}{r} z_n^{(\rho)}(kr) \frac{\partial}{\partial\theta} P_n^m(\cos\theta) \frac{\cos}{\sin} m\phi\, \hat{\boldsymbol{\theta}} \\ & \mp \frac{m}{r\sin\theta} z_n^{(\rho)}(kr) \quad P_n^m(\cos\theta) \frac{\sin}{\cos} m\phi\, \hat{\boldsymbol{\phi}} \end{aligned} \tag{3.11}$$

$$\begin{aligned} \mathbf{m}^{\rho}_{\substack{e\\o}mn} = {} & \mp \frac{m}{\sin\theta} z_n^{(\rho)}(kr) \quad P_n^m(\cos\theta) \frac{\sin}{\cos} m\phi\, \hat{\boldsymbol{\theta}} \\ & - z_n^{(\rho)}(kr) \frac{\partial}{\partial\theta} P_n^m(\cos\theta) \frac{\cos}{\sin} m\phi\, \hat{\boldsymbol{\phi}} \end{aligned} \tag{3.12}$$

$$\begin{aligned} \mathbf{n}^{\rho}_{\substack{e\\o}mn} = {} & \left[\frac{n(n+1)}{kr}\right] z_n^{(\rho)}(kr) \quad P_n^m(\cos\theta) \frac{\cos}{\sin} m\phi\, \hat{\mathbf{r}} \\ & + \frac{1}{kr}\frac{\partial}{\partial r}[r z_n^{(\rho)}(kr)] \frac{\partial}{\partial\theta} P_n^m(\cos\theta) \frac{\cos}{\sin} m\phi\, \hat{\boldsymbol{\theta}} \\ & \mp \frac{m}{kr\sin\theta}\frac{\partial}{\partial r}[r z_n^{(\rho)}(kr)] \quad P_n^m(\cos\theta) \frac{\sin}{\cos} m\phi\, \hat{\boldsymbol{\phi}}. \end{aligned} \tag{3.13}$$

The incident plane wave is now expanded in terms of the spherical vector wave functions. The relation between the rectangular and spherical coordinate systems is readily seen from Fig. 1. Thus,

$$\hat{\mathbf{i}}e^{-jkz} = e^{-jkr\cos\theta}(\sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}) \tag{3.14}$$

$$\hat{\mathbf{j}}e^{-jkz} = e^{-jkr\cos\theta}(\sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}). \tag{3.15}$$

Since $\hat{\mathbf{i}}e^{-jkz}$ and $\hat{\mathbf{j}}e^{-jkz}$ are solenoidal vectors, they may be expanded in terms of $\mathbf{m}_{\substack{e\\o}mn}$ and $\mathbf{n}_{\substack{e\\o}mn}$ alone. They are finite at $r = 0$; therefore their expansions require Bessel functions of the first kind. Also, their ϕ dependence requires that $m = 1$ in the expansions. Finally, comparison of the even and the odd properties of Eqs. (3.12) and (3.13) with Eqs. (3.14) and (3.15) shows that the formal expansions are

$$\hat{\mathbf{i}}e^{-jkr\cos\theta} = \sum_{n=0}^{\infty}(C_n^1\mathbf{m}_{o1n}^1 + C_n^2\mathbf{n}_{e1n}^1) \tag{3.16}$$

$$\hat{\mathbf{j}}e^{-jkr\cos\theta} = \sum_{n=0}^{\infty}(C_n^3\mathbf{m}_{e1n}^1 + C_n^4\mathbf{n}_{o1n}^1). \tag{3.17}$$

The coefficients are evaluated in the usual way by applying the orthogonality conditions. For example, to evaluate C_n^1, Eq. (3.16) is multiplied scalarly by $\sin\theta\,\mathbf{m}_{o1n'}^1$ and the resulting equation is integrated over θ and ϕ. This gives

$$\begin{aligned} & \int_0^{2\pi}\int_0^{\pi} \hat{\mathbf{i}}\cdot\mathbf{m}_{o1n'}^1 e^{-jkr\cos\theta}\sin\theta\,d\theta\,d\phi \\ & \qquad = \int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}(C_n^1\mathbf{m}_{o1n}^1 + C_n^2\mathbf{n}_{e1n}^1)\cdot\mathbf{m}_{o1n'}^1\sin\theta\,d\theta\,d\phi. \end{aligned} \tag{3.18}$$

Using the relations

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{m}_{e1n'}{}^1 \cdot \mathbf{n}_{e1n}{}^1 \sin\theta \, d\theta \, d\phi = 0 \tag{3.19}$$

$$\int_0^{2\pi}\int_0^{\pi} \overline{\mathbf{m}_{o1n}{}^1} \cdot \mathbf{m}_{o1n'}{}^1 \sin\theta \, d\theta \, d\phi = \begin{cases} 0 & \text{for } n' \neq n \\ \dfrac{2\pi[n(n+1)]^2 \, |z_n{}^{(1)}(kr)|^2}{2n+1} & \text{for } n' = n \end{cases} \tag{3.20}$$

$$\int_0^{2\pi}\int_0^{\pi} \hat{\mathbf{i}} \cdot \mathbf{m}_{o1n}{}^1 e^{-jkr\cos\theta} \sin\theta \, d\theta \, d\phi = 2\pi(-j)^n \, n(n+1) \, |z_n{}^{(1)}(kr)|^2, \tag{3.21}$$

it is seen that

$$C_n{}^1 = \left[\frac{2n+1}{n(n+1)}\right](-j)^n. \tag{3.22}$$

Similarly,

$$C_n{}^2 = -\left[\frac{2n+1}{n(n+1)}\right](-j)^{n+1} \tag{3.23}$$

$$C_n{}^3 = -\left[\frac{2n+1}{n(n+1)}\right](-j)^n = -C_n{}^1 \tag{3.24}$$

$$C_n{}^4 = -\left[\frac{2n+1}{n(n+1)}\right](-j)^{n+1} = C_n{}^2. \tag{3.25}$$

Therefore, the incident plane wave expansions are

$$\mathbf{E}_i = E_0 \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (\mathbf{m}_{o1n}{}^1 + j\mathbf{n}_{e1n}{}^1) \tag{3.26}$$

$$\mathbf{B}_i = -\frac{E_0}{v_0} \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (\mathbf{m}_{e1n}{}^1 - j\mathbf{n}_{o1n}{}^1). \tag{3.27}$$

The induced secondary field is constructed of two parts, one applying inside the sphere and the other applying at all points outside. These parts are written as expansions similar to those for the incident plane wave but with unknown amplitude coefficients. The part applying outside the sphere is referred to as the scattered field and is indicated by a subscript s. Since this field must be regular at infinity and must satisfy the radiation condition, Bessel functions of the second kind are required. The scattered field is written

$$\mathbf{E}_s = E_0 \sum_{n=1}^{r} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (a_n{}^r \mathbf{m}_{o1n}{}^3 + jb_n{}^r \mathbf{n}_{e1n}{}^3) \tag{3.28}$$

$$\mathbf{B}_s = -\frac{E_0}{v_0} \sum_{n=1}^{r} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (b_n{}^r \mathbf{m}_{e1n}{}^3 - ja_n{}^r \mathbf{n}_{o1n}{}^3). \tag{3.29}$$

The field inside the sphere is referred to as the transmitted field and it is indicated by a subscript t. Since this field is finite at $r = 0$, Bessel functions of the first kind are required. Also, the free space constants are replaced by the complex constants of the sphere. Thus, the complex wave number

$$k_1 = \omega(\mu_0 \xi)^{\frac{1}{2}} = \omega\left(\mu_0\epsilon_s - j\frac{\mu_0\sigma_s}{\omega}\right)^{\frac{1}{2}} \tag{3.30}$$

replaces k, and the complex velocity

$$v_1 = (\mu_0 \xi)^{-\frac{1}{2}} \tag{3.31}$$

replaces v_0. In Eq. (3.29), ϵ_e is the real effective absolute dielectric constant and σ_e is the real effective conductivity. The transmitted field is now written

$$\mathbf{E}_t = E_0\sum_{n=1}^{\infty}(-j)^n\left[\frac{2n+1}{n(n+1)}\right](a_n^t\mathbf{m}_{o1n}^1 + jb_n^t\mathbf{n}_{e1n}^1) \tag{3.32}$$

$$\mathbf{B}_t = -\frac{E_0}{v_1}\sum_{n=1}^{\infty}(-j)^n\left[\frac{2n+1}{n(n+1)}\right](b_n^t\mathbf{m}_{e1n}^1 - ja_n^t\mathbf{n}_{o1n}^1). \tag{3.33}$$

Equations (3.28) and (3.29) represent a formal solution for the scattered field. All that is needed to complete the formal solution is the evaluation of the amplitude coefficients a_n^s and b_n^s. This is done by applying the boundary conditions at the surface of the sphere. The boundary conditions at $r = a$ are *

$$\hat{r}\times(\mathbf{E}_i+\mathbf{E}_s) = \hat{r}\times\mathbf{E}_t \tag{3.34}$$

$$\hat{r}\times(\mathbf{B}_i+\mathbf{B}_s) = \hat{r}\times\mathbf{B}_t. \tag{3.35}$$

The ϕ component of Eq. (3.34) yields

$$\frac{P_n^1(\cos\theta)}{\sin\theta}\left\{z_n^{(1)}(\alpha)+a_n^s z_n^{(2)}(\alpha)-a_n^t z_n^{(1)}(N\alpha)\right\} + j\frac{dP_n^{(1)}(\cos\theta)}{d\theta}\left\{\frac{[\alpha z_n^{(1)}(\alpha)]'+b_n^s[\alpha z_n^{(2)}(\alpha)]'}{ka}-\frac{b_n^t[N\alpha z_n^{(1)}(N\alpha)]'}{k_1a}\right\}=0. \tag{3.36}$$

The θ component of Eq. (3.34) yields

$$\frac{dP_n^1(\cos\theta)}{d\theta}\left\{z_n^{(1)}(\alpha)+a_n^s z_n^{(2)}(\alpha)-a_n^t z_n^{(1)}(N\alpha)\right\} + j\frac{P_n^1(\cos\theta)}{\sin\theta}\left\{\frac{[\alpha z_n^{(1)}(\alpha)]'+b_n^s[\alpha z_n^{(2)}(\alpha)]'}{ka}-\frac{b_n^t[N\alpha z_n^{(1)}(N\alpha)]'}{k_1a}\right\}=0. \tag{3.37}$$

The ϕ component of Eq. (3.35) yields

$$\frac{P_n^1(\cos\theta)}{\sin\theta}\left\{\frac{z_n^{(1)}(\alpha)+b_n^s z_n^{(2)}(\alpha)}{v_0}-\frac{b_n^t z_n^{(1)}(N\alpha)}{v_1}\right\} + j\frac{dP_n^1(\cos\theta)}{d\theta}\left\{\frac{[\alpha z_n^{(1)}(\alpha)]'+a_n^s[\alpha z_n^{(2)}(\alpha)]'}{v_0ka}-\frac{a_n^t[N\alpha z_n^{(1)}(N\alpha)]'}{v_1k_1a}\right\}=0. \tag{3.38}$$

The θ component of Eq. (3.35) yields

$$\frac{dP_n^1(\cos\theta)}{d\theta}\left\{\frac{z_n^{(1)}(\alpha)+b_n^s z_n^{(2)}(\alpha)}{v_0}-\frac{b_n^t z_n^{(1)}(N\alpha)}{v_1}\right\} + j\frac{P_n^1(\cos\theta)}{\sin\theta}\left\{\frac{[\alpha z_n^{(1)}(\alpha)]'+a_n^s[\alpha z_n^{(2)}(\alpha)]'}{v_0ka}-\frac{a_n^t[N\alpha z_n^{(1)}(N\alpha)]'}{v_1k_1a}\right\}=0. \tag{3.39}$$

Equations (3.36), (3.37), (3.38) and (3.39) are all satisfied if the quantities in the braces { } are identically equal to zero. This yields

$$a_n^t z_n^{(1)}(N\alpha) - a_n^s z_n^{(2)}(\alpha) = z_n^{(1)}(\alpha) \tag{3.40}$$

* Equation (3.35) takes this simple form, without the magnetic constants, since both region 1 and region 2 are assumed to have the same permeability.

$$a_n^t[N\alpha z_n^{(1)}(N\alpha)]' - a_n^s[\alpha z_n^{(2)}(\alpha)]' = [\alpha z_n^{(1)}(\alpha)]' \tag{3.41}$$

$$Nb_n^t z_n^{(1)}(N\alpha) - b_n^s z_n^{(2)}(\alpha) = z_n^{(1)}(\alpha) \tag{3.42}$$

$$b_n^t[N\alpha z_n^{(1)}(N\alpha)]' - Nb_n^s[\alpha z_n^{(2)}(\alpha)]' = N[\alpha z_n^{(1)}(\alpha)]'. \tag{3.43}$$

These simultaneous equations are solved for the coefficients. Thus,

$$a_n^s = -\frac{z_n^{(1)}(N\alpha)[\alpha z_n^{(1)}(\alpha)]' - z_n^{(1)}(\alpha)[N\alpha z_n^{(1)}(N\alpha)]'}{z_n^{(1)}(N\alpha)[\alpha z_n^{(2)}(\alpha)]' - z_n^{(2)}(\alpha)[N\alpha z_n^{(1)}(N\alpha)]'} \tag{3.44}$$

$$b_n^s = -\frac{z_n^{(1)}(\alpha)[N\alpha z_n^{(1)}(N\alpha)]' - N^2 z_n^{(1)}(N\alpha)[\alpha z_n^{(1)}(\alpha)]'}{z_n^{(2)}(\alpha)[N\alpha z_n^{(1)}(N\alpha)]' - N^2 z_n^{(1)}(N\alpha)[\alpha z_n^{(2)}(\alpha)]'}. \tag{3.45}$$

Here the primes at the square brackets indicate differentiation with respect to the argument of the Bessel function inside the brackets. $\alpha = ka = 2\pi a/\lambda$; $N\alpha = k_1 a$; $N = k_1/k = (\xi/\epsilon_0)^{\frac{1}{2}}$ is the complex index of refraction for the sphere.

3.3. THE FAR-ZONE FIELD AND TOTAL POWER RELATIONS

To determine the scattered and absorbed power, it is necessary to know the far-zone scattered field — i.e., the field for $r \gg a$. This is found by introducing the asymptotic value for the spherical Hankel function

$$z_n^{(2)}(kr) \doteq \frac{1}{kr} e^{-j[kr - (n+1)\pi/2]} \tag{3.46}$$

into the expansion formulas for the scattered field, Eqs. (3.28) and (3.29), and neglecting terms of higher order than $(1/r)$. When this is done, it is seen that the far-zone scattered field in component form is

$$E_{sr} = B_{sr} = 0 \tag{3.47}$$

$$E_{s\theta} = \nu_0 B_{s\phi} = \left(\frac{j}{kr}\right) E_0 e^{-jkr} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(a_n^s \frac{P_n^1}{\sin\theta} + b_n^s \frac{dP_n^1}{d\theta}\right) \cos\phi \tag{3.48}$$

$$E_{s\phi} = -\nu_0 B_{s\theta} = -\left(\frac{j}{kr}\right) E_0 e^{-jkr} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(a_n^s \frac{dP_n^1}{d\theta} + b_n^s \frac{P_n^1}{\sin\theta}\right) \sin\phi. \tag{3.49}$$

The total field at any point external to the sphere is obtained by vector addition of the incident and scattered fields. Thus,

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s; \qquad \mathbf{B} = \mathbf{B}_i + \mathbf{B}_s. \tag{3.50}$$

The complex Poynting vector corresponding to this total field is defined by

$$\mathbf{S} = \frac{\nu_0}{2} \mathbf{E} \times \mathbf{B}^*. \tag{3.51}$$

The asterisk indicates the complex conjugate, and ν_0 is the magnetic constant of space ($\nu_0 = 1/\mu_0$). Since the far-zone field is transverse, the Poynting vector is radial. Substituting Eq. (3.50) into Eq. (3.51) and carrying out the vector product yields

$$\mathbf{S} = \hat{r}S_r = \hat{r}\left[\frac{\nu_0}{2}(E_{i\theta}B_{i\phi}^* - E_{i\phi}B_{i\theta}^*) + \frac{\nu_0}{2}(E_{s\theta}B_{s\phi}^* - E_{s\phi}B_{s\theta}^*) + \frac{\nu_0}{2}(E_{i\theta}B_{s\phi}^* + E_{s\theta}B_{i\phi}^* - E_{i\phi}B_{s\theta}^* - E_{s\phi}B_{i\theta}^*)\right]. \tag{3.52}$$

The complex Poynting vector corresponding to the incident field is

$$\mathbf{S}_i = \hat{r}S_{ir} = \hat{r}\frac{\nu_0}{2}(E_{i\theta}B_{i\phi}^* - E_{i\phi}B_{i\theta}^*), \tag{3.53}$$

and that corresponding to the scattered field is

$$\mathbf{S}_s = \hat{r}S_{sr} = \hat{r}\frac{\nu_0}{2}(E_{s\theta}B_{s\phi}^* - E_{s\phi}B_{s\theta}^*). \tag{3.54}$$

The total scattered power is obtained by integrating S_{sr} over the surface of a large sphere of radius r and taking the real part. Thus,

$$P_s = \frac{\nu_0}{2}\,\mathrm{Re}\int_0^{2\pi}\int_0^{\pi}(E_{s\theta}B_{s\phi}^* - E_{s\phi}B_{s\theta}^*)r^2\sin\theta\, d\theta\, d\phi. \tag{3.55}$$

The total power absorbed by the sphere is obtained by integrating S_r over the surface of a large sphere and taking the negative of the real part. Thus,

$$P_a = -\frac{\nu_0}{2}\,\mathrm{Re}\int_0^{2\pi}\int_0^{\pi}S_r r^2\sin\theta\, d\theta\, d\phi. \tag{3.56}$$

The total power removed from the incident wave is equal to the sum of P_s and P_a. Denote this by P_t. Since the integral of S_{ir} over a closed surface is zero, P_t can be expressed as an integral of the third term in Eq. (3.52). Thus,

$$P_t = P_s + P_a = -\frac{\nu_0}{2}\,\mathrm{Re}\int_0^{2\pi}\int_0^{\pi}(E_{i\theta}B_{s\phi}^* + E_{s\theta}B_{i\phi}^* - E_{i\phi}B_{s\theta}^* - E_{s\phi}B_{i\theta}^*)\, r^2\sin\theta\, d\theta\, d\phi. \tag{3.57}$$

The integral for P_s is evaluated as follows: From Eqs. (3.48) and (3.49)

$$\begin{aligned} E_{s\theta}B_{s\phi}^* = \frac{1}{\nu_0}\left(\frac{E_0}{kr}\right)^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg\{&\left[\frac{2n+1}{n(n+1)}\right]\cdot\left[\frac{2m+1}{m(m+1)}\right]\\ &\cdot\left[a_n^s a_m^{s*}\frac{P_n^1P_m^1}{\sin^2\theta} + b_n^s b_m^{s*}\frac{dP_n^1}{d\theta}\frac{dP_m^1}{d\theta} + a_n^s b_m^{s*}\frac{P_n^1}{\sin\theta}\frac{dP_m^1}{d\theta}\right.\\ &\left. + b_n^s a_m^{s*}\frac{dP_n^1}{d\theta}\frac{P_m^1}{\sin\theta}\right]\cos^2\phi\Bigg\} \end{aligned} \tag{3.58}$$

and

$$\begin{aligned} E_{s\phi}B_{s\theta}^* = \frac{-1}{\nu_0}\left(\frac{E_0}{kr}\right)^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg\{&\left[\frac{2n+1}{n(n+1)}\right]\cdot\left[\frac{2m+1}{m(m+1)}\right]\cdot\left[a_n^s a_m^{s*}\frac{dP_n^1}{d\theta}\frac{dP_m^1}{d\theta}\right.\\ &+ b_n^s b_m^{s*}\frac{P_n^1P_m^1}{\sin^2\theta} + a_n^s b_m^{s*}\frac{dP_n^1}{d\theta}\frac{P_m^1}{\sin\theta}\\ &\left. + b_n^s a_m^{s*}\frac{P_n^1}{\sin\theta}\frac{dP_m^1}{d\theta}\right]\sin^2\phi\Bigg\}. \end{aligned} \tag{3.59}$$

Substitute Eqs. (3.58) and (3.59) into Eq. (3.55) and recall that

$$\int_0^{2\pi} \cos^2\phi\, d\phi = \int_0^{2\pi} \sin^2\phi\, d\phi = \pi.$$

Then,

$$P_s = \frac{\pi \nu_0 E_0^2}{2\omega_0 k^2} \operatorname{Re} \int_0^{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \left[\frac{2n+1}{n(n+1)}\right] \cdot \left[\frac{2m+1}{m(m+1)}\right] \right.$$
$$\cdot \left[a_n^s a_m^{s*} \left(\frac{P_n^1 P_m^1}{\sin^2\theta} + \frac{dP_n^1}{d\theta}\frac{dP_m^1}{d\theta} \right) + b_n^s b_m^{s*} \left(\frac{P_n^1 P_m^1}{\sin^2\theta} + \frac{dP_n^1}{d\theta}\frac{dP_m^1}{d\theta} \right) \right.$$
$$+ a_n^s b_m^{s*} \left(\frac{P_n^1}{\sin\theta}\frac{dP_m^1}{d\theta} + \frac{dP_n^1}{d\theta}\frac{P_m^1}{\sin\theta} \right)$$
$$\left.\left. + b_n^s a_m^{s*} \left(\frac{dP_n^1}{d\theta}\frac{P_m^1}{\sin\theta} + \frac{P_n^1}{\sin\theta}\frac{dP_m^1}{d\theta} \right) \right] \sin\theta \right\} d\theta. \tag{3.60}$$

Using the following identities,

$$\int_0^{\pi} \left(\frac{P_n^1}{\sin\theta}\frac{dP_m^1}{d\theta} + \frac{dP_n^1}{d\theta}\frac{P_m^1}{\sin\theta} \right) \sin\theta\, d\theta = 0 \qquad \text{for all } n \text{ and } m \tag{3.61}$$

and

$$\int_0^{\pi} \left(\frac{dP_n^1}{d\theta}\frac{dP_m^1}{d\theta} + \frac{P_n^1 P_m^1}{\sin^2\theta} \right) \sin\theta\, d\theta = \begin{cases} 0 & \text{for } n \neq m \\ \dfrac{2[n(n+1)]^2}{(2n+1)} & \text{for } n = m, \end{cases} \tag{3.62}$$

one gets

$$P_s = \frac{\pi \nu_0 E_0^2}{\omega_0 k^2} \sum_{n=1}^{\infty} (2n+1)\ (a_n^s a_n^{s*} + b_n^s b_n^{s*}). \tag{3.63}$$

By a similar process,

$$P_t = -\frac{\pi \nu_0 E_0^2}{\omega_0 k^2} \operatorname{Re} \sum_{n=1}^{\infty} (2n+1)(a_n^s + b_n^s). \tag{3.64}$$

P_s and P_t may be expressed in terms of the cross sections defined in Section 2. Thus, using Eq. (2.1), the total scattering cross section is

$$Q_s = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1)(a_n^s a_n^{s*} + b_n^s b_n^{s*}), \tag{3.65}$$

and, using Eq. (2.3), the total attenuation cross section (including both absorption and scattering) is

$$Q_t = -\frac{2\pi}{k^2} \operatorname{Re} \sum_{n=1}^{\infty} (2n+1)(a_n^s + b_n^s). \tag{3.66}$$

3.4. THE BACK-SCATTERING CROSS SECTION

It is now possible to determine the back-scattering cross section defined in Section 2.2. From Eq. (2.9)

$$\sigma = \frac{8\pi\zeta_0}{E_0^2} \frac{dP_s}{d\Omega}\bigg|_{\theta=\pi}. \tag{3.67}$$

From Eq. (3.55)

$$\frac{dP_s}{d\Omega} = \frac{\nu_0 r^2}{2} (E_{s\theta} B_{s\phi}^* - E_{s\phi} B_{s\theta}^*). \tag{3.68}$$

Using Eqs. (3.48) and (3.49), and noting that

$$\lim_{\theta\to\pi}\frac{P_n^1(\cos\theta)}{\sin\theta} = -\lim_{\theta\to\pi}\frac{dP_n^1(\cos\theta)}{d\theta} = (-1)^{n+1}\left[\frac{n(n+1)}{2}\right], \tag{3.69}$$

it is seen that

$$\frac{dP_s}{d\Omega}\bigg|_{\theta=\pi} = \frac{\eta_0 E_0^2}{8\eta_0 k^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}(2n+1)(2m+1)(a_n^s - b_n^s)(a_m^{s*} - b_m^{s*}). \tag{3.70}$$

Therefore,

$$\sigma = \frac{\pi}{k^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}(2n+1)(2m+1)(a_n^s - b_n^s)(a_m^{s*} - b_m^{s*}). \tag{3.71}$$

An alternative method of deriving the equation for the back-scattering cross section is to use the equivalent definition of σ involving only the far-zone scattered field. In view of the orientation of the coordinate system used (see Fig. 1), Eq. (2.11) is

$$\sigma = 4\pi r^2\left|\frac{E_{s\phi}\left(\begin{smallmatrix}\theta=\pi\\ \phi=\pi/2\end{smallmatrix}\right)}{E_0}\right|^2 \tag{3.72}$$

or

$$\sigma = 4\pi r^2\left|\frac{E_{s\theta}\left(\begin{smallmatrix}\theta=\pi\\ \phi=0\end{smallmatrix}\right)}{E_0}\right|^2. \tag{3.73}$$

Using Eqs. (3.48) and (3.49) together with Eq. (3.69) yields

$$|E_{s\theta}|_{\substack{\theta=\pi\\ \phi=0}} = |E_{s\phi}|_{\substack{\theta=\pi\\ \phi=\pi/2}} = \left(\frac{E_0}{kr}\right)\left|\sum_{n=1}^{\infty}\left(\frac{2n+1}{2}\right)\left(-1\right)^n\left(a_n^s - b_n^s\right)\right| \tag{3.74}$$

and

$$\sigma = \left(\frac{\pi}{k^2}\right)\left|\sum_{n=1}^{\infty}\left(2n+1\right)\left(-1\right)^n\left(a_n^s - b_n^s\right)\right|^2. \tag{3.75}$$

Equation (3.75) is identical with Eq. (3.71).

3.5. THE SCATTERING AMPLITUDE COEFFICIENTS

From the preceding sections it is seen that the evaluation of the far-zone scattered field and the various cross sections reduces to the determination of the scattering amplitude coefficients, a_n^s and b_n^s, and the summation of the appropriate series. The scattered field may be regarded as due to the forced oscillation of magnetic and electric multipoles. If this point of view is taken, it is easy to see that the coefficients a_n^s are associated with oscillations of magnetic type, and the coefficients b_n^s are associated with oscillations of electric type. It is recalled (Eq. (3.12)) that the vector wave function $\mathbf{m}^{e}_{omn}$ has no radial component. Therefore, if the coefficients b_n^s are all zero and the coefficients a_n^s are not zero, the E field is purely transverse and the B field has a radial component. (See Eqs. (3.28) and (3.29).) Similarly, if the coefficients a_n^s are all zero and the coefficients b_n^s are not, the E_s field has a radial component and the B_s field is purely transverse. The magnetic and electric types of oscillation are sometimes called transverse electric (TE) and transverse magnetic (TM) types, respectively.

As stated above, the coefficients a_n^s and b_n^s may be associated with the forced oscillations of magnetic and electric multipoles. Whenever the forced frequency approaches one of the natural frequencies of vibration, a condition of resonance occurs. The natural frequencies of vibration are determined by setting the

denominators equal to zero in Eqs. (3.44) and (3.45). However, the natural frequencies are complex while the impressed frequency is always real. Therefore, no difficulties arise at resonance.

The problem of the numerical evaluation of the scattering amplitude coefficients will now be considered. This problem is solved formally by Eqs. (3.44) and (3.45). However, they are quite formidable, and attention is usually directed toward the special cases where certain simplifications are possible. These will be reviewed, following Stratton (1941), before the more difficult general problem is taken up.

For a metal sphere, $|N\alpha| \gg 1$ and the asymptotic expressions for the Bessel functions may be used. They are

$$z_n^{(1)}(N\alpha) \doteq \frac{1}{N\alpha}\cos\left[N\alpha - \frac{(n+1)\pi}{2}\right], \tag{3.76}$$

$$[N\alpha z_n^{(1)}(N\alpha)]' \doteq -\sin\left[N\alpha - \frac{(n+1)\pi}{2}\right]. \tag{3.77}$$

Substituting these expressions into Eqs. (3.44) and (3.45) yields

$$a_n^s \doteq -\left\{\frac{z_n^{(1)}(\alpha)}{z_n^{(3)}(\alpha)}\right\}, \tag{3.78}$$

$$b_n^s \doteq -\left\{\frac{[\alpha z_n^{(1)}(\alpha)]'}{[\alpha z_n^{(3)}(\alpha)]'}\right\}. \tag{3.79}$$

The final equations are exact for a perfectly conducting sphere.

For a large sphere, $\alpha \gg 1$, the asymptotic expressions may be used for all the Bessel functions involved in Eqs. (3.44) and (3.45). The result is

$$a_n^s \doteq (-j)^n e^{j\alpha}\left[\frac{\sin\Lambda - N\cos\Lambda\tan M}{1 + jN\tan M}\right], \tag{3.80}$$

$$b_n^s \doteq j^{n+1}e^{j\alpha}\left[\frac{\cos\Lambda\tan M - N\sin\Lambda}{\tan M - jN}\right], \tag{3.81}$$

where

$$\Lambda = \alpha - \frac{(n+1)\pi}{2}; \quad M = N\alpha - \frac{(n+1)\pi}{2}. \tag{3.82}$$

These equations are included because they illustrate the general oscillatory behavior of the scattering amplitude coefficients.

For a small sphere, $\alpha \ll 1$, it is possible to get a good approximation for the scattering amplitude coefficients by substituting the series expansions of the spherical Bessel and Hankel functions into Eqs. (3.44) and (3.45) and keeping only a few terms of the resulting series. This has been done by L. Goldstein (1945), who corrects the results of Stratton (1941). The series expansions of the spherical functions are

$$z_n^{(1)}(\alpha) = 2^n\alpha^n \sum_{m=0}^{\infty}\left[\frac{(-1)^m(n+m)!}{m!(2n+2m+1)!}\right]\alpha^{2m} \tag{3.83}$$

$$z_n^{(3)}(\alpha) = 2^n\alpha^n \sum_{m=0}^{\infty}\left[\frac{(-1)^m(n+m)!}{m!(2n+2m+1)!}\right]\alpha^{2m} + \frac{j}{2^n\alpha^{n+1}}\sum_{m=0}^{\infty}\left[\frac{(2n-2m)!}{m!(n-m)!}\right]\alpha^{2m}. \tag{3.84}$$

Substituting these expansions into Eqs. (3.44) and (3.45) and keeping only a few terms,

$$a_n^e \doteq -j2^{2n}\left[\frac{n!}{(2n+1)!}\right]^2\cdot\left[\frac{N^2-1}{2n+3}\right]\alpha^{2n+3}$$

$$\cdot\left[1+\alpha^2\left\{\left[\frac{N^2-1}{2n+1}\right]-\left[\frac{N^2+1}{2(2n+5)}\right]\right\}+\dots\right] \tag{3.85}$$

$$b_n^e \doteq -j2^{2n}\left[\frac{n!}{(2n+1)!}\right]^2\cdot\left[\frac{(2n+1)(n+1)(N^2-1)}{(nN^2+n+1)}\right]\cdot\alpha^{2n+1}$$

$$\cdot\left[1+\alpha^2\left\{\frac{(2n+1)[(2n-1)N^2-n-1]}{(2n+3)(2n-1)(nN^2+n+1)}\right\}+\dots\right.$$

$$\left.-j2^{2n}\left\{\frac{n!}{(2n+1)!}\right\}^2\cdot\left\{\frac{(2n+1)(n+1)(N^2-1)}{(nN^2+n+1)}\right\}\alpha^{2n+1}+\dots\right]. \tag{3.86}$$

If α is so small that powers of α higher than the sixth may be neglected, only three coefficients are needed:

$$a_1^e \doteq \frac{-j}{45}(N^2-1)\alpha^5 \tag{3.87}$$

$$b_1^e \doteq \frac{-2j}{3}\left[\frac{N^2-1}{N^2+2}\right]\alpha^3\cdot\left\{1-\left[\frac{3(N^2-2)}{5(N^2+2)}\right]\alpha^2-\left[\frac{2j(N^2-1)}{3(N^2+2)}\right]\alpha^3\right\} \tag{3.88}$$

$$b_2^e \doteq \left[\frac{-j(N^2-1)}{15(2N^2+3)}\right]\alpha^5. \tag{3.89}$$

This is an important case since it applies to the raindrop problem for many of the radars in use. Equations (3.87), (3.88) and (3.89) are the same ones used by Ryde (1941, 1944) if account is taken of the difference in definition between the coefficients defined by Stratton (used here) and those of Mie (used by Ryde). The relation between the two sets of definitions is given by Goldstein (1945):

$$p_n^{\mathrm{Mie}} = (-1)^n j(2n+1)a_n^e \tag{3.90}$$

$$a_n^{\mathrm{Mie}} = (-1)^{n+1} j(2n+1)b_n^e. \tag{3.91}$$

It should be noted that if α is so small that powers of α higher than the third may be neglected, only the electric dipole mode is needed. This is the case of Rayleigh scattering.

It is seen that even with the simplifications afforded by the approximations involved in the special cases, the numerical evaluation of the scattering amplitude coefficients is quite involved. An exact evaluation would be even more so. Goldstein (1945) remarks, "An exact computation of these coefficients is out of the question on account of the lack of tables of Bessel and Hankel functions of complex argument needed here." The special cases discussed, however, do not cover all the problems of interest, so that it is necessary to consider the problem of getting reliable answers in the general case.

Although the evaluation of the scattering amplitude coefficients is difficult in the general case, it can be accomplished. In their study of the effects of meteorological elements on microwave radiation, Ryde and Ryde (1945) set up a schedule for computing the coefficients directly by calculating the values of the spherical Bessel and Hankel functions for the arguments needed and substituting them into the Mie equivalent of Eqs. (3.44) and (3.45). Since the work by Ryde and Ryde, Lowan (1949) has computed the coefficients for various values of α and six values of N corresponding to water in the microwave region. There are still no

extensive tables available, however, for the spherical functions of complex arguments, and there is no general method for getting exact values of the coefficients except the "brute force" method. In Section 3.6, such a method will be given.

3.6. EVALUATION OF THE SCATTERING AMPLITUDE COEFFICIENTS USING LOGARITHMIC DERIVATIVE FUNCTIONS

The general solution for the scattering amplitude coefficients is given by Eqs. (3.44) and (3.45). These equations may be rearranged as follows:

$$a_n' = -\left[\frac{z_n^{(1)}(\alpha)}{z_n^{(2)}(\alpha)}\right]\left\{\frac{\left(\frac{[\alpha z_n^{(1)}(\alpha)]'}{[\alpha z_n^{(1)}(\alpha)]}\right) - N\left(\frac{[N\alpha z_n^{(1)}(N\alpha)]'}{[N\alpha z_n^{(1)}(N\alpha)]}\right)}{\left(\frac{[\alpha z_n^{(2)}(\alpha)]'}{[\alpha z_n^{(2)}(\alpha)]}\right) - N\left(\frac{[N\alpha z_n^{(1)}(N\alpha)]'}{[N\alpha z_n^{(1)}(N\alpha)]}\right)}\right\} \tag{3.92}$$

$$b_n' = -\left[\frac{z_n^{(1)}(\alpha)}{z_n^{(2)}(\alpha)}\right]\left\{\frac{\left(\frac{[N\alpha z_n^{(1)}(N\alpha)]'}{[N\alpha z_n^{(1)}(N\alpha)]}\right) - N\left(\frac{[\alpha z_n^{(1)}(\alpha)]'}{[\alpha z_n^{(1)}(\alpha)]}\right)}{\left(\frac{[N\alpha z_n^{(1)}(N\alpha)]'}{[N\alpha z_n^{(1)}(N\alpha)]}\right) - N\left(\frac{[\alpha z_n^{(2)}(\alpha)]'}{[\alpha z_n^{(2)}(\alpha)]}\right)}\right\}. \tag{3.93}$$

It should be noted that in this form the equations involve the logarithmic derivatives of $[\alpha z_n^{(1)}(\alpha)]$, $[N\alpha z_n^{(1)}(N\alpha)]$ and $[\alpha z_n^{(2)}(\alpha)]$. Logarithmic derivatives of this type were used by Infeld (1947) in his study of the spherical antenna with a gap, and by Smith (1948) and Tai (1949) in the study of biconical antennas. If the notation of Smith and Tai is used, the Bessel density functions are defined

$$S_n(x) = [x^{\frac{1}{2}} J_{n+\frac{1}{2}}(x)], \tag{3.94}$$

$$R_n(x) = [x^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(x)]. \tag{3.95}$$

Following Tai (1949), the logarithmic derivatives of these Bessel density functions are defined as new functions, $\sigma_n(x)$ and $\rho_n(x)$. Thus,

$$\sigma_n(x) = \frac{S'_n(x)}{S_n(x)} = \frac{[xz_n^{(1)}(x)]'}{[xz_n^{(1)}(x)]}, \tag{3.96}$$

$$\rho_n(x) = \frac{R'_n(x)}{R_n(x)} = \frac{[xz_n^{(2)}(x)]'}{[xz_n^{(2)}(x)]}. \tag{3.97}$$

In terms of these functions, the scattering amplitude coefficients are

$$a_n' = -\left[\frac{z_n^{(1)}(\alpha)}{z_n^{(2)}(\alpha)}\right]\cdot\left[\frac{\sigma_n(\alpha) - N\sigma_n(N\alpha)}{\rho_n(\alpha) - N\sigma_n(N\alpha)}\right] \tag{3.98}$$

$$b_n' = -\left[\frac{z_n^{(1)}(\alpha)}{z_n^{(2)}(\alpha)}\right]\cdot\left[\frac{\sigma_n(N\alpha) - N\sigma_n(\alpha)}{\sigma_n(N\alpha) - N\rho_n(\alpha)}\right]. \tag{3.99}$$

This form appears to be simpler since there are fewer functions and no derivatives of functions involved explicitly. To make the apparent simplification real, however, it is necessary to demonstrate that values for the logarithmic derivative functions can be easily obtained. This will now be done.

Since the following derivations hold for all the Bessel density functions, involving either the Bessel, Neumann or Hankel functions, it is convenient to let

$$Z_n(x) = \begin{Bmatrix} J_n(x) \\ N_n(x) \\ H_n(x) \end{Bmatrix} = \text{the general cylindrical function,} \tag{3.100}$$

$$D_n(x) = x^{\frac{1}{2}} Z_{n+\frac{1}{2}}(x) = \text{the general Bessel density function and} \tag{3.101}$$

$$\delta_n(x) = \frac{D'_n(x)}{D_n(x)} = \text{the general logarithmic derivative function.} \tag{3.102}$$

Differentiate Eq. (3.101) with respect to x:

$$D'_n(x) = x^{\frac{1}{2}} Z'_{n+\frac{1}{2}}(x) + \tfrac{1}{2} x^{-\frac{1}{2}} Z_{n+\frac{1}{2}}(x). \tag{3.103}$$

Substitute the recurrence formula*

$$Z'_{n+\frac{1}{2}}(x) = Z_{n-\frac{1}{2}}(x) - \left[\frac{n+\frac{1}{2}}{x}\right] Z_{n+\frac{1}{2}}(x) \tag{3.104}$$

into Eq. (3.103) to eliminate the derivative of Z:

$$D'_n(x) = x^{\frac{1}{2}} \left[Z_{n-\frac{1}{2}}(x) - \frac{n}{x} Z_{n+\frac{1}{2}}(x) \right]. \tag{3.105}$$

Divide by $D_n(x)$:

$$\delta_n(x) = \frac{D'_n(x)}{D_n(x)} = \left[\frac{Z_{n-\frac{1}{2}}(x)}{Z_{n+\frac{1}{2}}(x)}\right] - \frac{n}{x}. \tag{3.106}$$

This is the standard equation used to evaluate the logarithmic derivative functions when x is real.

The difficulty comes when x is complex. Equation (3.106) cannot be used then because tables of $Z_{n+\frac{1}{2}}(x)$ are not available for x complex. In this case, use is made of the recurrence formula due to Infeld, (1947), which will now be derived. From Eq. (3.101),

$$Z_{n+\frac{1}{2}}(x) = x^{-\frac{1}{2}} D_n(x). \tag{3.107}$$

Differentiate:

$$Z'_{n+\frac{1}{2}}(x) = x^{-\frac{1}{2}} \left[D'_n(x) - \left(\frac{1}{2x}\right) D_n(x)\right]. \tag{3.108}$$

Differentiate again:

$$Z''_{n+\frac{1}{2}}(x) = x^{-\frac{1}{2}}\left[D''_n(x) - \frac{1}{x} D'_n(x) + \left(\frac{3}{4x^2}\right) D_n(x)\right]. \tag{3.109}$$

The Bessel equation of half-integer order is

$$x^2 Z''_{n+\frac{1}{2}}(x) + x Z'_{n+\frac{1}{2}}(x) + [x^2 - (n+\tfrac{1}{2})^2] Z_{n+\frac{1}{2}}(x) = 0. \tag{3.110}$$

Substitute Eqs. (3.107), (3.108) and (3.109) into Eq. (3.110) and collect terms.

$$D''_n(x) = \left[\frac{n(n+1)}{x^2} - 1\right] D_n(x). \tag{3.111}$$

This is the same as Eq. (D.1) of Infeld if n is replaced by $p - \frac{1}{2}$. It is now necessary to get an expression for $D'_n(x)$. Substitute Eqs. (3.107) and (3.108) into the recurrence formula*

* See, for example, Watson, *A Treatise on the Theory of Bessel Functions*, pp. 45 and 66.

$$Z_{n+\frac{1}{2}}(x) = \left[\frac{n-\frac{1}{2}}{x}\right] Z_{n-\frac{1}{2}}(x) - Z'_{n-\frac{1}{2}}(x) \tag{3.112}$$

and collect terms:

$$D_n(x) = \frac{n}{x} D_{n-1}(x) - D'_{n-1}(x)\cdot \tag{3.113}$$

Differentiate:

$$D'_n(x) = \frac{n}{x} D'_{n-1}(x) - \frac{n}{x^2} D_{n-1}(x) - D''_{n-1}(x)\cdot \tag{3.114}$$

Substitute Eq. (3.111) into Eq. (3.114) and collect terms:

$$D'_n(x) = \frac{n}{x} D'_{n-1}(x) - \frac{n^2}{x^2} D_{n-1}(x) + D_{n-1}(x)\cdot \tag{3.115}$$

Now, divide Eq. (3.115) by Eq. (3.113). Then,

$$\delta_n(x) = \frac{D'_n(x)}{D_n(x)} = \frac{\frac{n}{x} D'_{n-1}(x) - \frac{n^2}{x^2} D_{n-1}(x) + D_{n-1}(x)}{\frac{n}{x} D_{n-1}(x) - D'_{n-1}(x)}. \tag{3.116}$$

But,

$$\left[\frac{D'_{n-1}(x)}{D_{n-1}(x)}\right] = \delta_{n-1}(x)\cdot$$

Therefore,

$$\delta_n(x) = \frac{\frac{n}{x}\delta_{n-1}(x) - \frac{n^2}{x^2} + 1}{\frac{n}{x} - \delta_{n-1}(x)}. \tag{3.117}$$

This is the same as Eq. (D.6) of Infeld if n is replaced by $p - \frac{1}{2}$. It is sometimes easier to use Eq. (3.117) if both numerator and denominator are cleared of fractions. Thus,

$$\delta_n(x) = \frac{x^2 + nx\delta_{n-1}(x) - n^2}{nx - x^2\delta_{n-1}(x)}. \tag{3.118}$$

Equation (3.118) gives the recurrence relation whereby the logarithmic derivative function of any order can be found if the function of the next lower order is known. It will now be shown that the lowest order can be computed directly. Since $\sigma_n(x)$ is the only function with complex argument in Eqs. (3.98) and (3.99), only this function will be considered:

$$S_0(x) = [x^{\frac{1}{2}} J_{\frac{1}{2}}(x)] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin x, \tag{3.119}$$

$$S'_0(x) = [x^{\frac{1}{2}} J_{\frac{1}{2}}(x)]' = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos x. \tag{3.120}$$

Therefore,

$$\sigma_0(x) = \frac{S'_0(x)}{S_0(x)} = \cot x. \tag{3.121}$$

Now, if $x = c - jd$, then,

$$\sigma_0(x) = \cot(c - jd) = \frac{\sin 2c + j \sinh 2d}{\cosh 2d - \cos 2c}. \tag{3.122}$$

Higher-order terms may now be found using Eq. (3.118).

It is seen that by using Eqs. (3.98) and (3.99) together with Eqs. (3.106), (3.118) and (3.122), the scattering amplitude coefficients of any order can be computed exactly, even in regions where the spherical Bessel functions of complex arguments are not tabulated. In the next section this method is applied to the evaluation of the back-scattering cross section for one particular case involving water spheres with sizes comparable to the wavelength. This case has also been studied experimentally and comparison between the theoretical and experimental results is made in Section 4.

3.7. NUMERICAL COMPUTATIONS

The method given in Section 3.6 was used to compute the scattering amplitude coefficients for water spheres at $\lambda = 16.230$ cm and $0.6 < \alpha < 6$. These coefficients were then used to compute the back-scattering cross section, using Eq. (3.75). For plotting purposes, it is convenient to normalize the back-scattering cross section with respect to the geometrical cross section. If this is done, Eq. (3.75) becomes

$$\frac{\sigma}{\pi a^2} = \frac{1}{\alpha^2}\left| \sum_{n=1}^{\infty} (2n+1)\,(-1)^n\,(a_n{}^s - b_n{}^s)\right|^2 . \tag{3.123}$$

This is the equation actually used in the computations. Since the values of $a_n{}^s$ and $b_n{}^s$ decrease rapidly for $n > \alpha$, it is necessary to carry the summation only to $n \cong 2\alpha$.

Since the computed values of the back-scattering cross section depend on the index of refraction, it is important to have reliable values for this quantity. It is recalled that with $\mu_1 = \mu_2$, the complex index of refraction is related to the complex dielectric factor by

$$N = \left(\frac{\xi}{\epsilon_0}\right)^{\frac{1}{2}} = (\xi_r)^{\frac{1}{2}} = (\xi'_r - j\xi''_r)^{\frac{1}{2}}, \tag{3.124}$$

where ξ_r is the relative dielectric factor, and ξ'_r and $-\xi''_r$ are the real and imaginary parts of ξ_r. Ryde (1941, 1944) and Goldstein (1945) determined ξ_r by using the Debye (1929) formula,

$$\xi_r = \xi_{r0} + \frac{\xi_{r_\infty} - \xi_{r0}}{1 + j(\lambda_s/\lambda)}, \tag{3.125}$$

where ξ_{r0} is the relative optical dielectric factor; $\xi_{r\infty}$ is the relative static dielectric factor; and λ_s is the transition wavelength. They determined λ_s from Eq. (3.125) by using Collie's (1944) measured values of ξ'_r and ξ''_r for $\lambda = 1.26$ cm. The validity of using the Debye equations for the frequency dependence of ξ_r at constant temperature has been verified by Saxton (1945) and Collie et al. (1948). Slight differences have been found in the experimental values of λ_s, but in general the agreement is good. This same method has accordingly been used in the present research. The result is $\xi_r = 81 - j7.8$ at $\lambda = 16.230$ and temperatures near 20° C.

Some of the computed values for the logarithmic derivative functions and scattering amplitude coefficients are plotted in Figs. 2–19.

Figures 2 and 3 show $\sigma_n(\alpha)$ versus α for $n = 1, 2, 3, \ldots, 12$. The values of $\sigma_n(\alpha)$ were computed using Eq. (3.106) with $Z_{n\pm\frac{1}{2}}(\alpha) = J_{n\pm\frac{1}{2}}(\alpha)$. It is seen that $\sigma_n(\alpha)$ is a smooth, non-oscillating function. It approaches $+\infty$ as $(n+1)/\alpha$ when α approaches zero. It has poles at the zeros of $J_{n+\frac{1}{2}}(\alpha)$. The poles of $\sigma_n(\alpha)$, however, do not cause the scattering amplitude coefficients to blow up since it is always the product $J_{n+\frac{1}{2}}(\alpha)\cdot\sigma_n(\alpha)$ which appears in Eqs. (3.98) and (3.99), and this product has no poles.

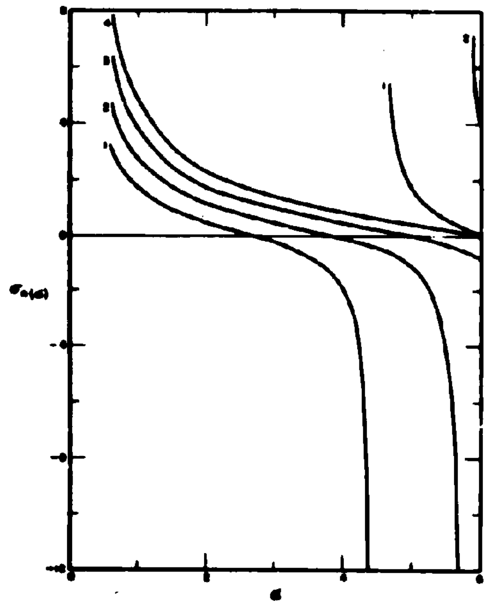

Fig. 2. The σ_n function for real argument.

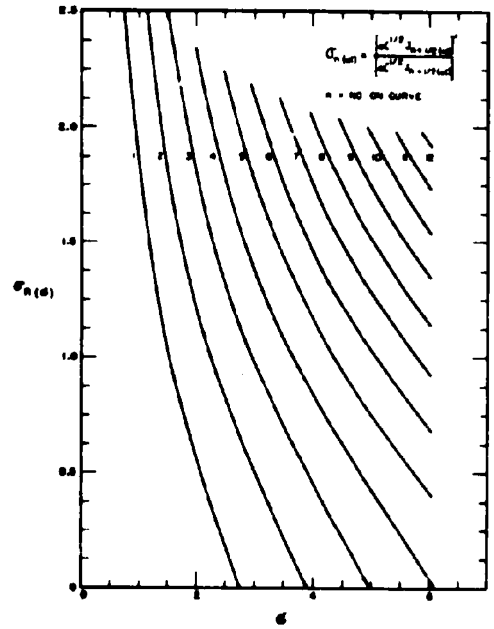

Fig. 3. The σ_n function for real argument.

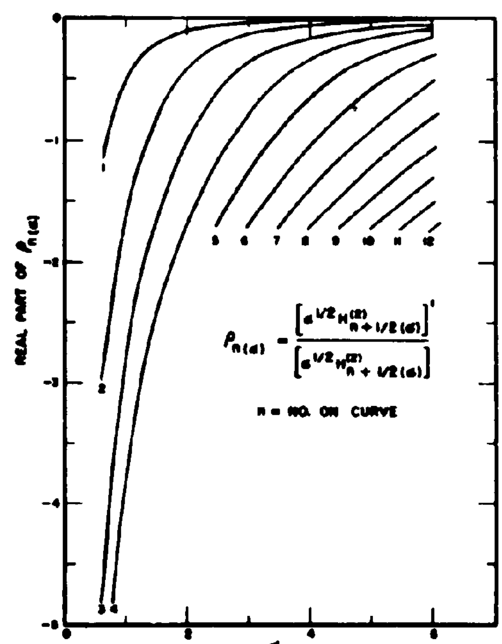

Fig. 4. Real part of the ρ_n function.

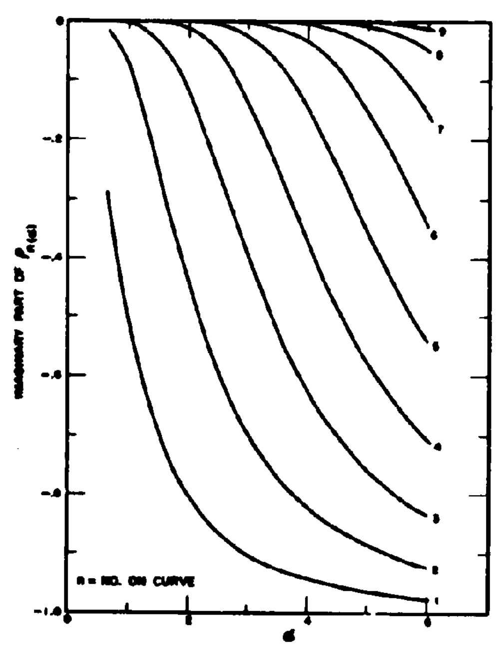

Fig. 5. Imaginary part of the ρ_n function.

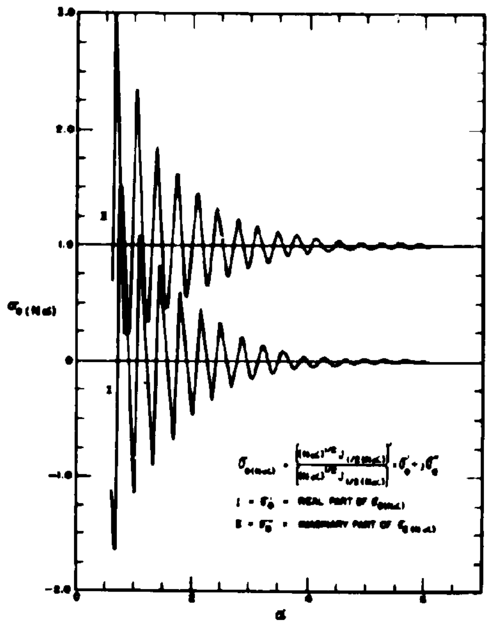

FIG. 6. The σ_0 function for complex argument.

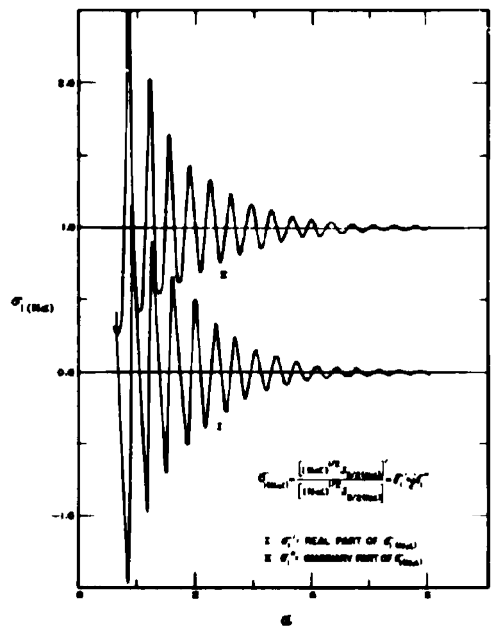

FIG. 7. The σ_1 function for complex argument.

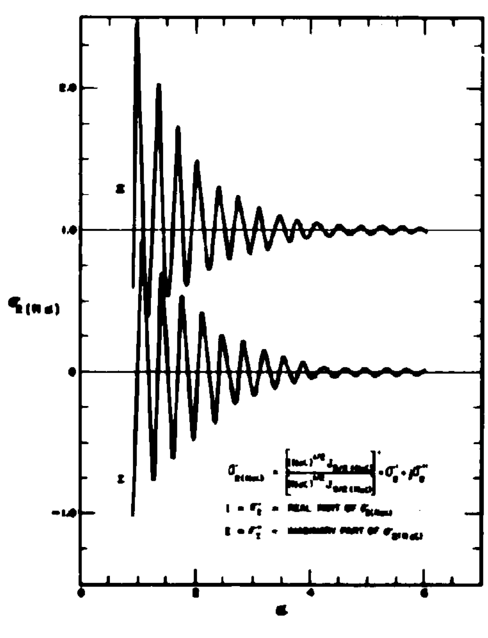

FIG. 8. The σ_2 function for complex argument.

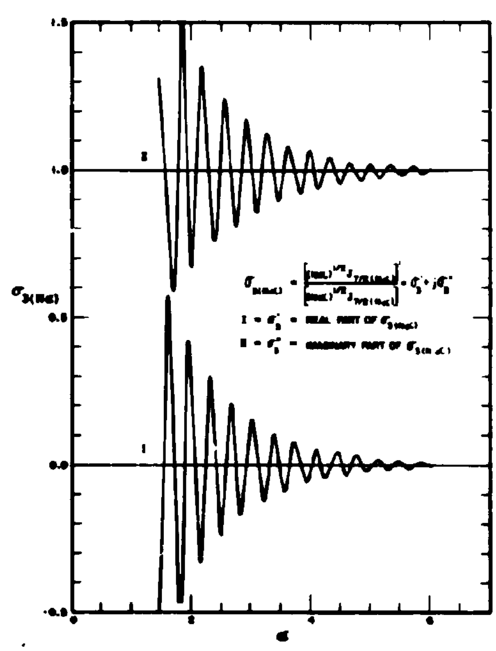

FIG. 9. The σ_3 function for complex argument.

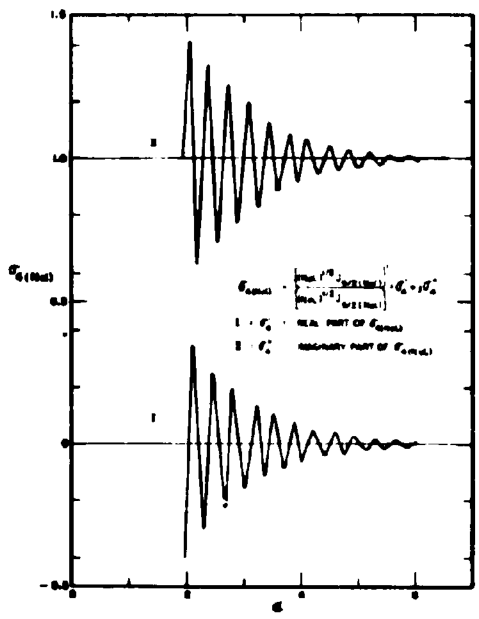

FIG. 10. The σ_4 function for complex argument.

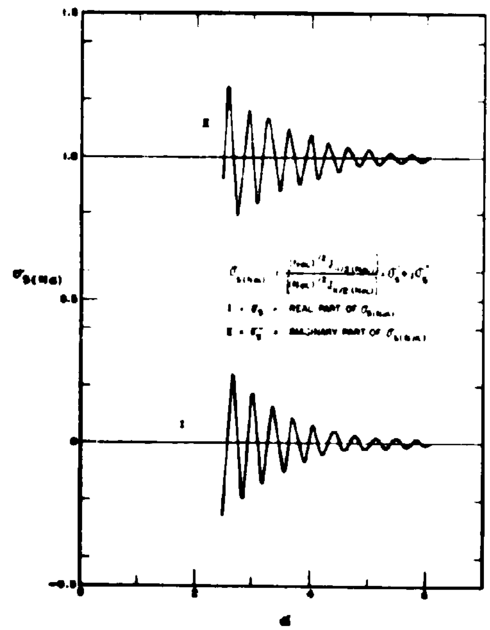

FIG. 11. The σ_5 function for complex argument.

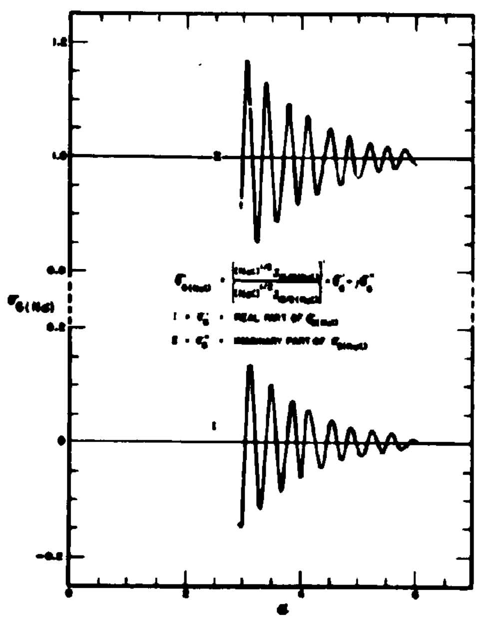

FIG. 12. The σ_6 function for complex argument.

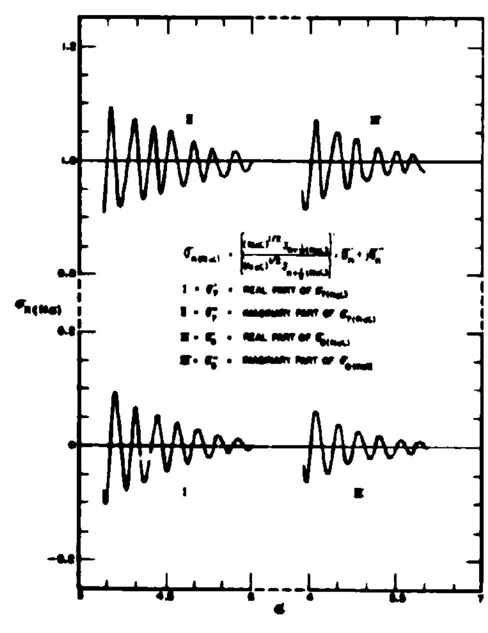

FIG. 13. The $\sigma_{7,8}$ function for complex argument.

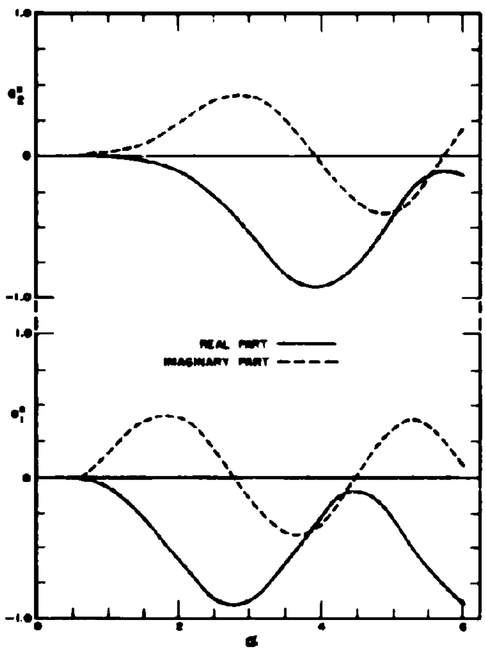

FIG. 14. The scattering amplitude coefficients, a_1^0 and a_2^0.

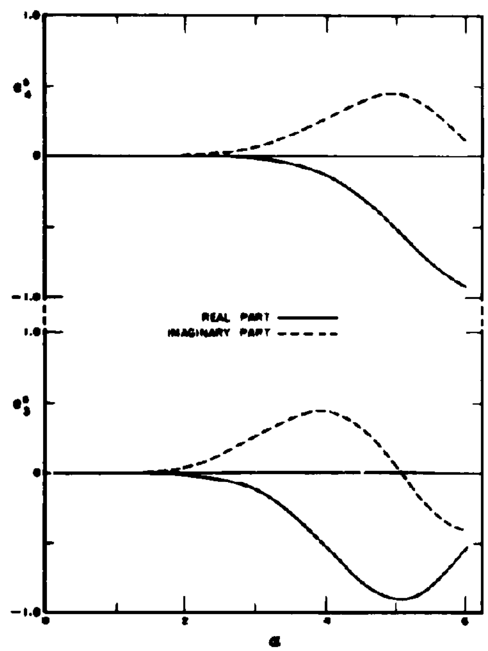

FIG. 15. The scattering amplitude coefficients, a_3^0 and a_4^0.

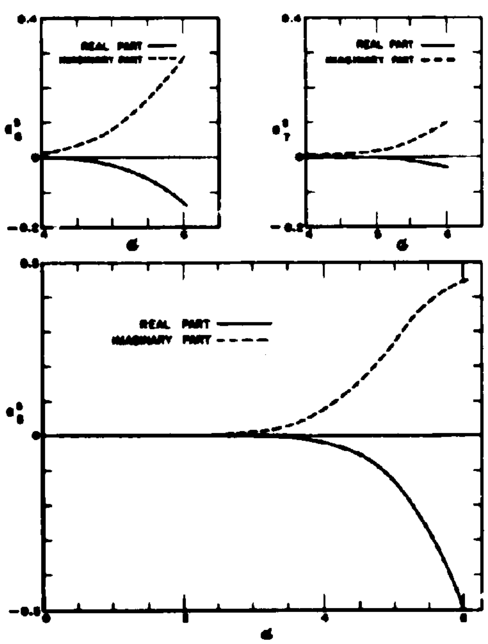

FIG. 16. The scattering amplitude coefficients, a_5^0, a_6^0 and a_7^0.

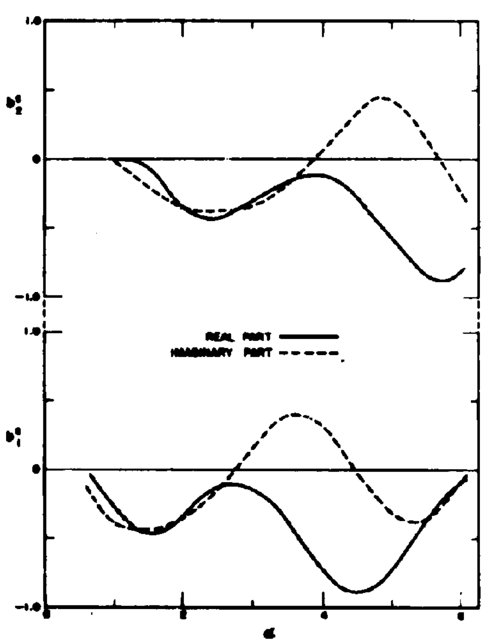

FIG. 17. The scattering amplitude coefficients, b_1^0 and b_2^0.

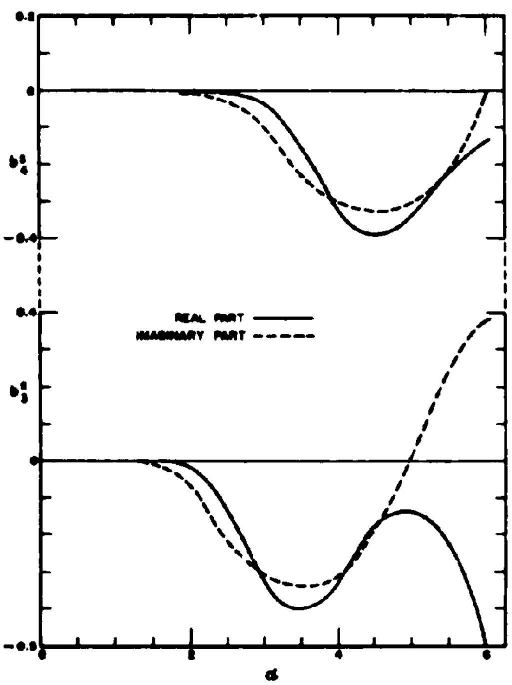

FIG. 18. The scattering amplitude coefficients, b_3' and b_4'.

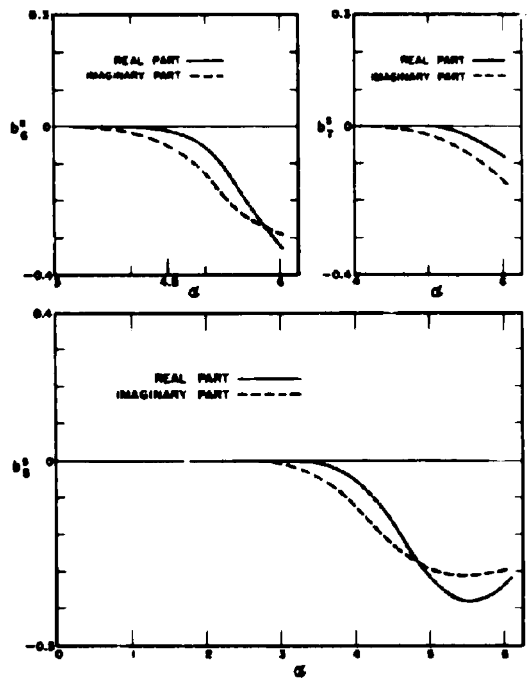

FIG. 19. The scattering amplitude coefficients, b_5', b_6' and b_7'.

Figures 4 and 5 show the real and imaginary parts of $\rho_n(\alpha)$. The values of these functions were computed using Eq. (3.106) with $Z_{n\pm\frac{1}{2}}(\alpha) = H_{n\pm\frac{1}{2}}{}^{(2)}(\alpha)$. It is seen that they are also non-oscillatory. The real part of $\rho_n(\alpha)$ approaches $-\infty$ as $-n/\alpha$ when α approaches zero. As α increases, the real part increases rapidly from $-\infty$ and approaches zero. The imaginary part starts at zero and approaches -1 as α increases.

Figures 6 to 13 show the real and imaginary parts of $\sigma(N\alpha)$ for $n = 1, 2, \ldots, 8$. It is seen that these are rapidly oscillating, highly damped functions. The real part approaches zero and the imaginary part approaches 1 as α increases.

Figures 14 to 19 show the scattering amplitude coefficients a_n' and b_n', for $n = 1, 2, \ldots, 7$. The general oscillatory behavior of the coefficients is seen, as well as the rapid decrease in amplitude for $n > \alpha$.

The theoretical normalized back-scattering cross-section curve is shown in Fig. 20, where it is compared with the one obtained experimentally.

4. EXPERIMENTAL DETERMINATION OF THE BACK-SCATTERING FROM WATER SPHERES AND COMPARISON WITH THEORY

4.1. INTRODUCTION

Since the introduction of microwave techniques to the field of meteorology, much work has been done on the specific applications to the detection of reflections from rain and other hydrometeors. Unfortunately, most of this work has been qualitative in nature. Some quantitative measurements on rain have been

made by H. Goldstein (1945), Marshall et al. (1947), Langille and Gunn (1948) and Hooper and Kippax (1950). Considering the number of variables involved, these measurements show good agreement with theory. It is difficult, however, to isolate the individual effects when there are many variables, and it seems desirable to have some laboratory measurements on single spheres. Recently, this need has been at least partially filled by the work of Klotzbaugh and Duckett (1949) on plastic spheres simulating raindrops and the work of Aden (1950) on actual water spheres. In Sections 4.2 to 4.4, the techniques used by Aden will be described briefly, and the results obtained will be compared with those determined from theory.

4.2. THE EXPERIMENTAL PROBLEM

The experimental measurement of the reflection of electromagnetic waves from individual water spheres involves two important difficulties: (1) Since the sphere is a very low gain reflector, it is hard to obtain reliable answers by ordinary pulsing techniques and (2) it is not easy to maintain a water sphere while measurements are being made. For these reasons, no direct laboratory measurements had been undertaken successfully prior to the work mentioned above. Even now, these difficulties almost preclude the taking of measurements in the region of greatest interests — i.e., the spectral region where water spheres of actual raindrop size are comparable to the wavelength. However, it is possible to perform an experiment at slightly longer wavelengths and to increase the size of the water spheres to get into the critical size region. This is the experiment that actually was performed. It is believed that this technique is useful since comparison between experiment and theory at one frequency should give a good indication of the correlation to be expected at other frequencies.

4.3. THE EXPERIMENTAL TECHNIQUES

The experimental method used to measure the back-scattering cross section was the standing-wave method of D. King (1948). This method utilizes an image screen technique, and the standing waves set up on the screen by the interaction of the incident and reradiated waves are measured along the radial line between the scatterer and the source. This method offers the advantages of a system having absolute calibration, relatively simple equipment and measurements at low power. The main disadvantages are that it requires an image screen which is large, uniform and rigid, and that obtaining results is very time consuming. The derivation of the formulas needed and a discussion of the approximations, as well as a detailed description of the equipment used, are given by Aden (1950).

One of the worst obstacles to making measurements on water spheres is the inability to maintain such spheres while measurements are being taken. This obstacle was overcome by using as a container thin hemispherical shell forms of Styrofoam* mounted on aluminum disks. Since Styrofoam has dielectric properties extremely close to those of air, it had a negligible effect on the measurements. Thus, when a form was filled with water and the disk was inserted into its proper place in the image screen, the effect was that of a hemisphere of water exposed over a large image screen. By image theory, the measurements taken were the same as for a complete sphere in free space.

4.4. COMPARISON OF THE THEORETICAL AND EXPERIMENTAL RESULTS

Measurements of the back-scattering cross section were made on 30 water spheres in the electrical size region $0.74 \leq \alpha \leq 5.90$. In Fig. 20, the results of these measurements are plotted, together with those

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* Product of The Dow Chemical Co., Midland, Mich.

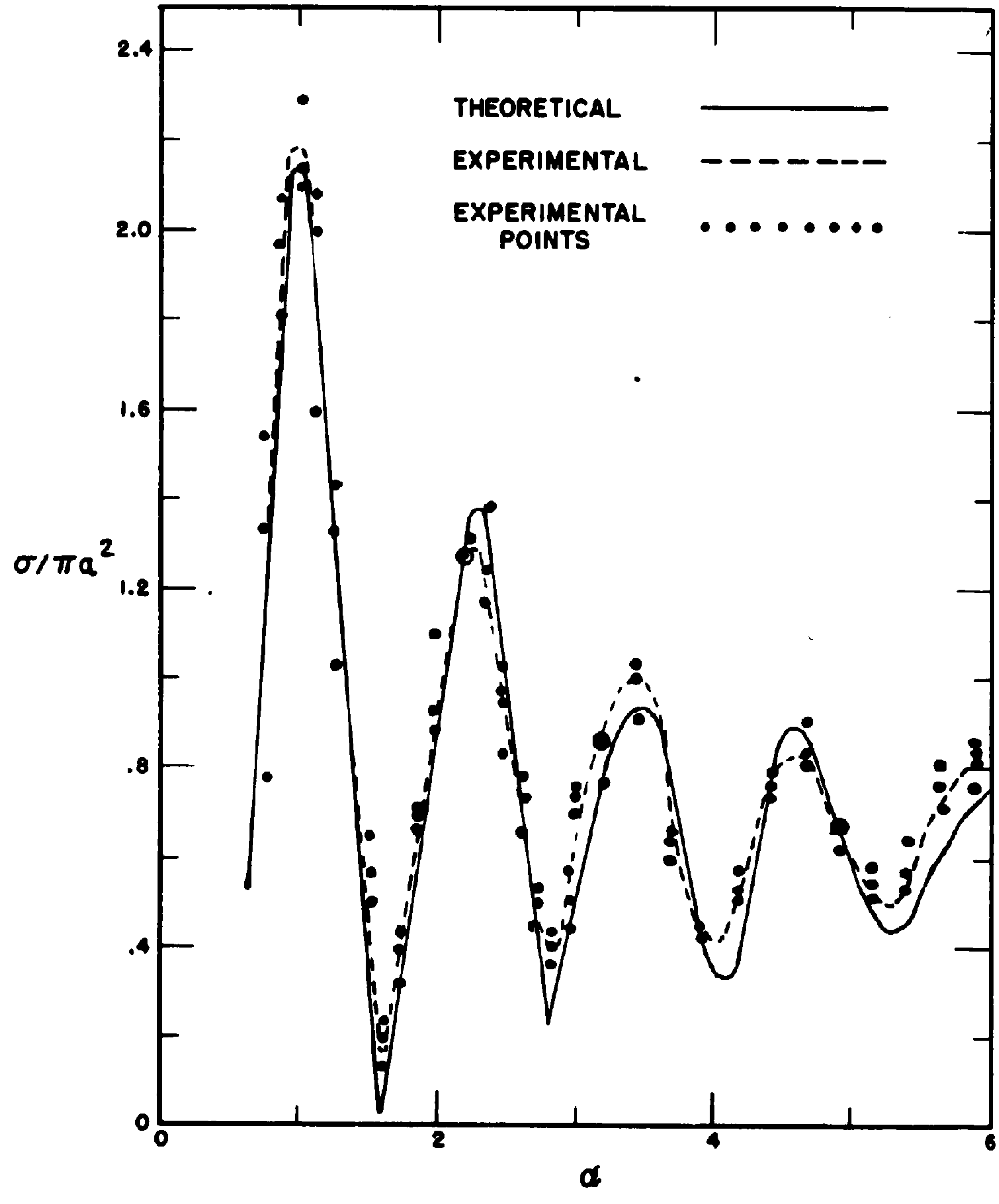

FIG. 20. Back scattering from water spheres.

determined from theory. The theoretical curve was obtained by using the method of logarithmic derivative functions given in Section 3. It is seen that there is very good agreement between the experimental and theoretical results. For spheres very much smaller in size than those considered here, the results should be given to a good approximation by the Rayleigh law. For spheres of larger size than those considered here, the results should approach the large-size approximation of about 0.64.

5. THE THEORY OF THE SCATTERING OF A PLANE WAVE BY A SPHERE WITH A CONCENTRIC SPHERICAL SHELL

5.1. INTRODUCTION

In the preceding sections, the scattering of electromagnetic radiation by a single sphere has been considered, and its application to the more general problem of microwave reflection from rain has been indicated. Another physical problem of interest is the scattering of microwave radiation by melting snow and ice particles. For purposes of analysis, this physical problem may, to a first approximation, be replaced by the mathematical problem of finding the reflection from a sphere with a concentric spherical shell of different dielectric factor. This latter problem can be solved rigorously.

The method of solution is a direct extension of that used for the problem of the scattering from a single sphere. The incident plane wave is expanded in terms of the orthogonal spherical vector wave functions of Stratton (1941). The induced secondary fields in the various regions are written as similar expansions with unknown amplitude coefficients. As in the previous case, the unknown coefficients are determined from the boundary conditions. In this case, however, the matching must be done simultaneously over two boundary surfaces instead of one.

5.2. FORMULATION AND SOLUTION OF THE PROBLEM

Consider a sphere of radius a, complex dielectric factor ξ_1 and permeability μ_0 which is surrounded by a spherical shell of inner radius a and outer radius b with a complex dielectric factor ξ_2 and permeability μ_0. The resulting configuration is assumed to be isolated in free space. The interior of the sphere, the interior of the shell and the surrounding space are called regions 1, 2 and 3, respectively. As in Section 2, the center of the sphere is chosen as the origin of a rectangular coordinate system; the incident electromagnetic wave is propagated along the z axis, and the electric vector is linearly polarized parallel to the x direction. This is illustrated in Fig. 21.

With the conditions stated above, the expressions for the incident plane wave are exactly the same as in Section 3:

$$\mathbf{E}_i = \hat{\mathbf{i}}E_x = \hat{\mathbf{i}}E_0 e^{-jkz} = E_0 \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (\mathbf{m}_{o1n}^1 + j\mathbf{n}_{e1n}^1) \tag{5.1}$$

$$\mathbf{B}_i = \hat{\mathbf{j}}B_y = \hat{\mathbf{j}}\frac{E_0}{v_0} e^{-jkz} = -\frac{E_0}{v_0} \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (\mathbf{m}_{e1n}^1 - j\mathbf{n}_{o1n}^1). \tag{5.2}$$

The induced secondary field must now be constructed in three parts, one applying in each of the three regions defined above. These parts are written as expansions similar to those for the incident wave but

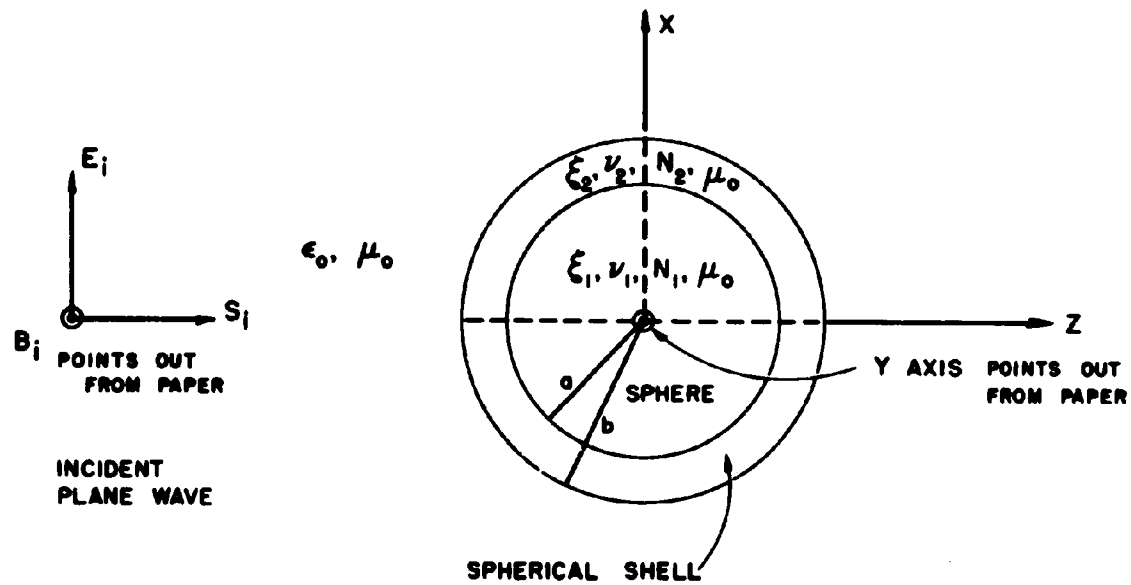

FIG. 21. Plane wave incident upon a sphere with a concentric spherical shell.

with unknown amplitude coefficients. The part applying outside the shell is again referred to as the scattered field and is indicated by a subscript s. The form of this expansion is the same as in Eqs. (3.28) and (3.29) although the values of a^s and b^s are different here:

$$\mathbf{E}_s = E_0 \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (a_n^s \mathbf{m}_{o1n}^3 + j b_n^s \mathbf{n}_{e1n}^3) \tag{5.3}$$

$$\mathbf{B}_s = -\frac{E_0}{v_0} \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (b_n^s \mathbf{m}_{e1n}^3 - j a_n^s \mathbf{n}_{o1n}^3). \tag{5.4}$$

By analogy with the previous problem, the field inside the sphere is called the transmitted field, and it is indicated by a subscript t. Again, the formal expansions are the same as before (Eqs. (3.32) and (3.33)) with the understanding that a_n^t and b_n^t have different values:

$$\mathbf{E}_t = E_0 \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (a_n^t \mathbf{m}_{o1n}^1 + j b_n^t \mathbf{n}_{e1n}^1) \tag{5.5}$$

$$\mathbf{B}_t = -\frac{E_0}{v_1} \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] (b_n^t \mathbf{m}_{e1n}^1 - j a_n^t \mathbf{n}_{o1n}^1). \tag{5.6}$$

Here, $v_1 = (\mu_0 \xi_1)^{-\frac{1}{2}}$ is the complex characteristic velocity of the sphere.

Inside the spherical shell, the terms involving Bessel functions of both first and second kinds must be retained. The formal expansions may be written

$$\mathbf{E}_2 = E_0 \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] [a_n^2 \mathbf{m}_{o1n}^1 + A_n^2 \mathbf{m}_{o1n}^2 + j(b_n^2 \mathbf{n}_{e1n}^1 + B_n^2 \mathbf{n}_{e1n}^2)] \tag{5.7}$$

$$\mathbf{B}_2 = -\frac{E_2}{v_2} \sum_{n=1}^{\infty} (-j)^n \left[\frac{2n+1}{n(n+1)}\right] [b_n^2 \mathbf{m}_{e1n}^1 + B_n^2 \mathbf{m}_{e1n}^2 - j(a_n^2 \mathbf{n}_{o1n}^1 + A_n^2 \mathbf{n}_{o1n}^2)] \tag{5.8}$$

where a_n^2, A_n^2, b_n^2, B_n^2 are unknown amplitude coefficients, and $v_2 = (\mu_0\xi_2)^{-1}$ is the complex characteristic velocity of the shell.

Equations (5.3) through (5.8) represent a formal solution for the induced secondary field. All that is needed to complete the formal solution is the evaluation of the eight amplitude coefficients. This is done by applying the boundary conditions at the two surfaces of dielectric discontinuity. The boundary conditions at $r = a$ are

$$\hat{r} \times \mathbf{E}_2\Big|_{r=a} = \hat{r} \times E_i\Big|_{r=a} \tag{5.9}$$

$$\hat{r} \times \mathbf{B}_2\Big|_{r=a} = \hat{r} \times B_i\Big|_{r=a} \tag{5.10}$$

and at $r = b$ are

$$\hat{r} \times (\mathbf{E}_i + \mathbf{E}_s)\Big|_{r=b} = \hat{r} \times \mathbf{E}_2\Big|_{r=b} \tag{5.11}$$

$$\hat{r} \times (\mathbf{B}_i + \mathbf{B}_s)\Big|_{r=b} = \hat{r} \times \mathbf{B}_2\Big|_{r=b} . \tag{5.12}$$

These lead to two sets of simultaneous equations each involving four unknowns as follows:

$$-\frac{[N_1\alpha z_n^{(1)}(N_1\alpha)]'}{N_1}\; b_n^i + \frac{[N_2\alpha z_n^{(1)}(N_2\alpha)]'}{N_2}\, b_n^2 + \frac{[N_2\alpha z_n^{(2)}(N_2\alpha)]'}{N_2}\; B_n^2 = 0 \tag{5.13}$$

$$-N_1 z_n^{(1)}(N_1\alpha)\; b_n^i + N_2 z_n^{(1)}(N_2\alpha)\; b_n^2 + N_2 z_n^{(2)}(N_2\alpha)\; B_n^2 = 0 \tag{5.14}$$

$$-[\nu z_n^{(2)}(\nu)]' b_n^s + 0\;\; b_n^i + \frac{[N_2\nu z_n^{(1)}(N_2\nu)]'}{N_2}\; b_n^2 + \frac{[N_2\nu z_n^{(2)}(N_2\nu)]'}{N_2}\; B_n^2 = [\nu z_n^{(1)}(\nu)]' \tag{5.15}$$

$$-z_n^{(2)}(\nu) b_n^s + 0\;\; b_n^i + N_2 z_n^{(1)}(N_2\nu)\; b_n^2 + N_2 z_n^{(2)}(N_2\nu)\; B_n^2 = z_n^{(1)}(\nu) \tag{5.16}$$

and

$$-z_n^{(1)}(N_1\alpha)\; a_n^i + z_n^{(1)}(N_2\alpha)\; a_n^2 + z_n^{(2)}(N_2\alpha)\; A_n^2 = 0 \tag{5.17}$$

$$-[N_1\alpha z_n^{(1)}(N_1\alpha)]'\; a_n^i + [N_2\alpha z_n^{(1)}(N_2\alpha)]'\, a_n^2 + [N_2\alpha z_n^{(2)}(N_2\alpha)]'\, A_n^2 = 0 \tag{5.18}$$

$$-z_n^{(2)}(\nu)\;\; a_n^s + 0\;\; a_n^i + z_n^{(1)}(N_2\nu)\;\; a_n^2 + z_n^{(2)}(N_2\nu)\;\; A_n^2 = z_n^{(1)}(\nu) \tag{5.19}$$

$$-[\nu z_n^{(2)}(\nu)]'\;\; a_n^s + 0\;\; a_n^i + [N_2\nu z_n^{(1)}(N_2\nu)]'\, a_n^2 + [N_2\nu z_n^{(2)}(N_2\nu)]'\, A_n^2 = [\nu z_n^{(1)}(\nu)]'. \tag{5.20}$$

Here, $\alpha = 2\pi a/\lambda$, $\nu = 2\pi b/\lambda$; $N_1 = \sqrt{\xi_1/\epsilon_0}$ and $N_2 = \sqrt{\xi_2/\epsilon_0}$ are the complex indices of refraction for the sphere and shell, respectively. As before, the primes at the square brackets indicate differentiation with respect to the argument of the Bessel function inside the brackets. These two sets of simultaneous equations may be solved for the eight amplitude coefficients.

To evaluate the back-scattering cross section, it is necessary only to solve for the scattering amplitude coefficients, a_n^s and b_n^s. If this is done, and if the derivatives of the Bessel density functions are eliminated by introducing the logarithmic derivative functions, the result is:

$$b_n^s = -\frac{z_n^{(1)}(\nu)}{z_n^{(2)}(\nu)} \cdot \left[\frac{F_1 + \sigma_n(\nu) G_1}{F_1 + \rho_n(\nu) G_1}\right] \tag{5.21}$$

$$a_n^s = -\frac{z_n^{(1)}(\nu)}{z_n^{(2)}(\nu)} \cdot \left[\frac{F_2 + \sigma_n(\nu) G_2}{F_2 + \rho_n(\nu) G_2}\right] \tag{5.22}$$

where

$$F_1 = N_2 z_n^{(1)}(N_1\alpha)\sigma_n(N_1\alpha)[z_n^{(1)}(N_2\alpha)z_n^{(2)}(N_2\nu)\rho_n(N_2\nu) - z_n^{(1)}(N_2\nu)\sigma_n(N_2\nu)z_n^{(2)}(N_2\alpha)]$$
$$+ N_1 z_n^{(1)}(N_1\alpha)[z_n^{(1)}(N_2\nu)\sigma_n(N_2\nu)z_n^{(2)}(N_2\alpha)\rho_n(N_2\alpha) - z_n^{(1)}(N_2\alpha)\sigma_n(N_2\alpha)z_n^{(2)}(N_2\nu)\rho_n(N_2\nu)] \quad (5.23)$$

$$G_1 = N_2^2 z_n^{(1)}(N_1\alpha)\sigma_n(N_1\alpha)[z_n^{(1)}(N_2\nu)z_n^{(2)}(N_2\alpha) - z_n^{(1)}(N_2\alpha)z_n^{(2)}(N_2\nu)]$$
$$+ N_1N_2 z_n^{(1)}(N_1\alpha)[z_n^{(1)}(N_2\alpha)\sigma_n(N_2\alpha)z_n^{(2)}(N_2\nu) - z_n^{(1)}(N_2\nu)z_n^{(2)}(N_2\alpha)\rho_n(N_2\alpha)] \quad (5.24)$$

$$F_2 = N_2^2 z_n^{(1)}(N_1\alpha)[z_n^{(1)}(N_2\nu)\sigma_n(N_2\nu)z_n^{(2)}(N_2\alpha)\rho_n(N_2\alpha) - z_n^{(1)}(N_2\alpha)\sigma_n(N_2\alpha)z_n^{(2)}(N_2\nu)\rho_n(N_2\nu)]$$
$$+ N_1N_2 z_n^{(1)}(N_1\alpha)\sigma_n(N_1\alpha)[z_n^{(1)}(N_2\alpha)z_n^{(2)}(N_2\nu)\rho_n(N_2\nu) - z_n^{(1)}(N_2\nu)\sigma_n(N_2\nu)z_n^{(2)}(N_2\alpha)] \quad (5.25)$$

$$G_2 = N_2 z_n^{(1)}(N_1\alpha)[z_n^{(1)}(N_2\alpha)\sigma_n(N_2\alpha)z_n^{(2)}(N_2\nu) - z_n^{(1)}(N_2\nu)z_n^{(2)}(N_2\alpha)\rho_n(N_2\alpha)]$$
$$+ N_1 z_n^{(1)}(N_1\alpha)\sigma_n(N_1\alpha)[z_n^{(1)}(N_2\nu)z_n^{(2)}(N_2\alpha) - z_n^{(1)}(N_2\alpha)z_n^{(2)}(N_2\nu)]. \quad (5.26)$$

The equation for the back-scattering cross section is the same as before:

$$\sigma = \frac{\lambda^2}{4\pi}\left|\sum_{n=1}^{\infty}(-1)^n(2n+1)(a_n' - b_n')\right|^2. \quad (5.27)$$

Equation (5.27), together with Eqs. (5.21) through (5.26), gives the formal solution to the problem of back scattering from a sphere with a concentric spherical shell.* As a check, it should be noted that this solution should reduce to that for a single sphere if $\alpha = \nu$ and $N_1 = N_2 = N$. Since the equation for the back-scattering cross section is the same in both cases, this requires the scattering amplitude coefficients to be the same. Examination of Eqs. (5.21) through (5.26) reveals that they do reduce to Eqs. (3.98) and (3.99) under the conditions stated.

Although the application of this problem to particular physical problems has been indicated, the details of these applications will not be given here. They are, however, being given attention by other members of the Geophysics Research Division.

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* Since the completion of this manuscript Aden and Kerker (1951) have treated the more general case where all three regions have complex parameters.

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